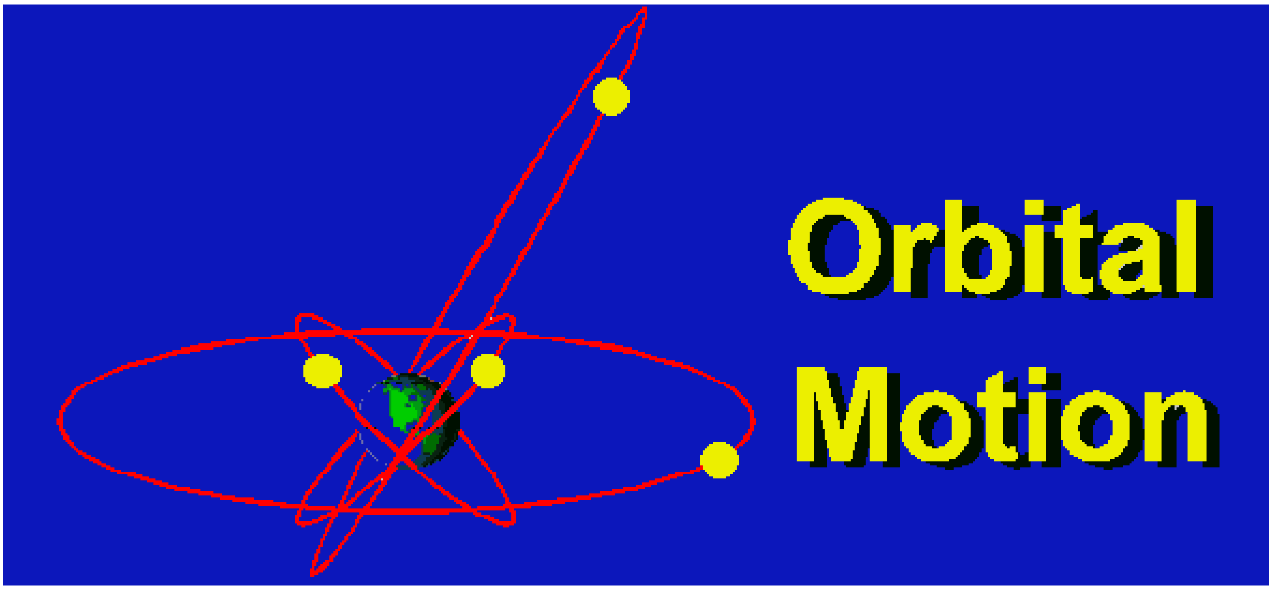


Advanced Satellite Consulting Ltd

Alex da Silva Curiel
20 April 2000

Author's Disclaimer

Table of Contents

1 GLOSSARY OF SYMBOLS

a	semi-major axis
b	semi-minor axis
e / **e**	eccentricity / eccentricity vector
E	eccentric anomaly
G	universal gravitational constant
h / **h**	angular momentum (per unit mass) / angular momentum vector
i	orbital inclination
I	unit vector along the direction to the vernal equinox
J	unit vector in the orbital plane perpendicular to **I**
J_n	zonal harmonic coefficient
K	unit vector normal to orbital plane
M	mean anomaly
M_e	mass of the earth
n	mean motion (mean angular velocity)
n	**K**x**h**, vector along the line from earth centre of gravity to ascending node
r	radius vector from earth centre of gravity towards satellite
R	disturbing potential
R_e	mean equatorial radius of earth
S	acceleration component along the radius vector
T	kinetic energy per unit mass
T	acceleration component in the orbital plane perpendicular to **S**
t	time
u	argument of latitude, $u = \omega + \nu$
v	velocity
V	potential energy per unit mass
W	acceleration component normal to the orbital plane
α	right ascension
β	flight path angle
δ	declination
ε	total energy (per unit mass)
φ	
λ	geodesic longitude
	gravitational parameter; for earth satellites: $= GM$
ν	
θ_g	Greenwich mean sidereal time

SOME USEFUL CONSTANTS

General

6.668 10^{-11} $^{-1}$ m sec^{-2}

shown:

1 inch =
1 foot = 0.3048 m
1,609.344 m

1 nautical mile =

2.2

1 mean solar second (smoothed) = 1 ephemeris second (to about 1 part in 10)
length of a tropical year at 1900.0 =
1 mean solar second = 1.002
1 mean sidereal second = 0.997 269 566 4 mean solar seconds
24 ^{h} ^{min} 56 .555 4 of mean sidereal time =

1 mean sidereal day = 23 56^{min} ^{s}.090 54 of mean solar time =
86,164.090 54 mean solar seconds

Length of a Tropical Year
365.242 20 mean solar days

Length of a Julian Year =
Greenwich Mean Sidereal Time (GMST) at 0^{h}
6^{h} ^{m} 45 .836 + 86 401 84^{s} $_{u}$ + 0 .0929 T_u
where T_u ^{h} UT on

2.3

Parameter	Symbol		Unit	Source
	a	6378 137.0		WGS 1984
Polar Radius		6356 752.3	m	
Flattening	f			WGS 1984
Mass	$_{e}$	5.9742 10	kg	IAU System
orbit		1.49597870 10 (=1 Astronomical Unit)	km	
Mean Eccentricity	e			IAU System
Mean Obliquity of the	i	23 26' 21''.448		IAU System
	g	9.80621 - 0.0259 cos 2 (φ	ms^{-2}	
Gravitational Parameter	or GM_e	^{14}	m /sec^{2}	
Geopotential coefficient		1082.626 10^{-6}		
Geostationary Orbit		42164.16	km	
		6.6107	Earth Radii	

2.4 Moon

Parameter	Symbol	Value	Unit	Source
Mass	M_{moon}	$7.3483\ 10^{22}$	kg	IAU System
Semi-major axis of orbit		384 401	km	IAU System
Eccentricity	e	0.054 900		IAU System
Mean Inclination to Ecliptic	i	$5^{o}.145396$	deg	IAU System
Surface gravity	g	1.62	ms^{-2}	IAU System
Gravitational Parameter	μ or GM_{moon}	$4.92105\ 10^{12}$	m^3/sec^2	

2.5 Sun

Parameter	Symbol	Value	Unit	Source
Mass	M_{sun}	$1.9891\ 10^{30}$	kg	IAU System
Solar Constant (intensity of solar radiation at mean earth distance)	I	$1.36\ 10^{3}$	Joule/m^2/ sec	
Surface gravity	g	$2.74\ 10^{2}$	ms^{-2}	IAU System
Gravitational Parameter	μ or GM_{sun}	$1.32712438\ 10^{20}$	m^3/sec^2	IAU System

3 GLOSSARY OF EXPRESSIONS

Words in *italics* are defined elsewhere in this glossary

3.1 Apoapsis

The point of an elliptical orbit that is farthest away from the gravitational centre of the system consisting of the primary body and the satellite (In Earth-based systems, the apoapsis is called *apogee*).

3.2 Apogee

The point in the satellite orbit that is farthest from the gravitational centre of the earth.

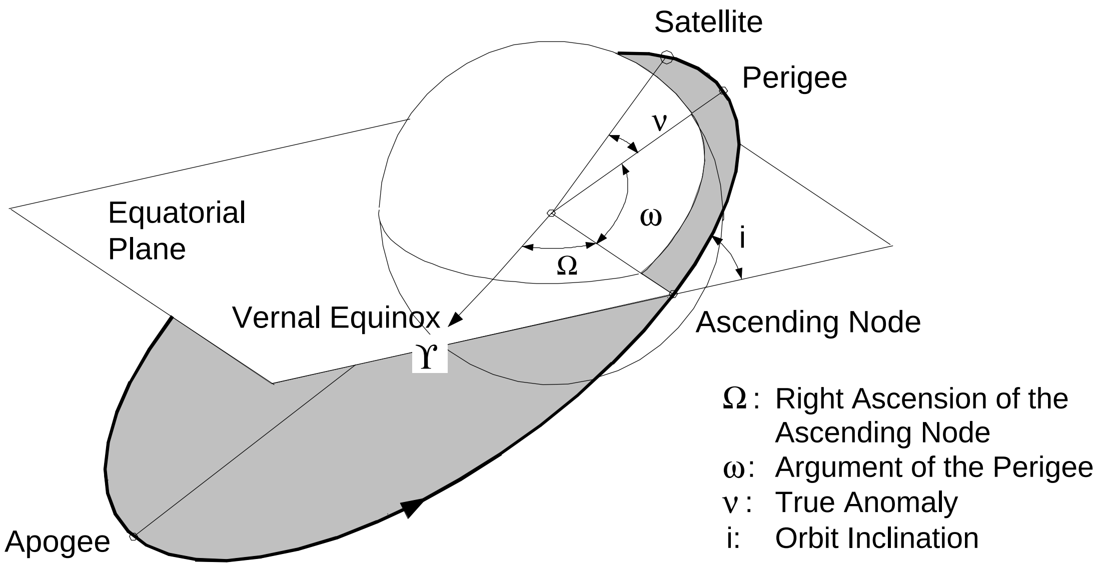


Figure 3-1 Satellite Orbit in Space

3.3 Apogee Altitude, or Apogee Height

The altitude of *apogee* above a specified reference point serving to represent the surface of the earth.

3.4 Apogee Kick Motor (AKM)

Motor, used once during the lifetime of a geostationary satellite to provide the large *delta-v* required to turn a highly elliptical orbit with *apogee* at the geostationary altitude into a circular, *geostationary orbit*. Apogee Kick Motors are needed because many launchers are not able deliver a satellite into geostationary orbit (the Russian Proton launcher is an exception to this rule).

3.5 Apsis

One of the extreme end-points of the *major axis* of an elliptical orbit (*apogee* and *perigee* are apses).

3.6 Argument of the Perigee (ARGP)

Angle in the plane of the satellite orbit between the *ascending node* and the *perigee*, measured in the direction of the satellite motion (see Figure 3-1 Satellite Orbit in Space).

3.7 Ascending Node

Point in the equatorial plane where the satellite crosses through the equatorial plane in a northerly direction (see Figure 3-1 Satellite Orbit in Space).

3.8 Atmospheric Drag

Slowing down force acting on a satellite due to the Earth's atmosphere. Below 160 km height, the atmosphere causes a satellite orbit to decay (spiral down) within a few revolutions. Drag is the predominant force affecting satellite lifetime in low earth orbit. Above 700 km, drag has hardly any influence.

3.9 Declination

Angle in a meridian, measured northward from the *ecliptic* to a particular direction (direction defined as the line from the earth centre to a particular celestial object).

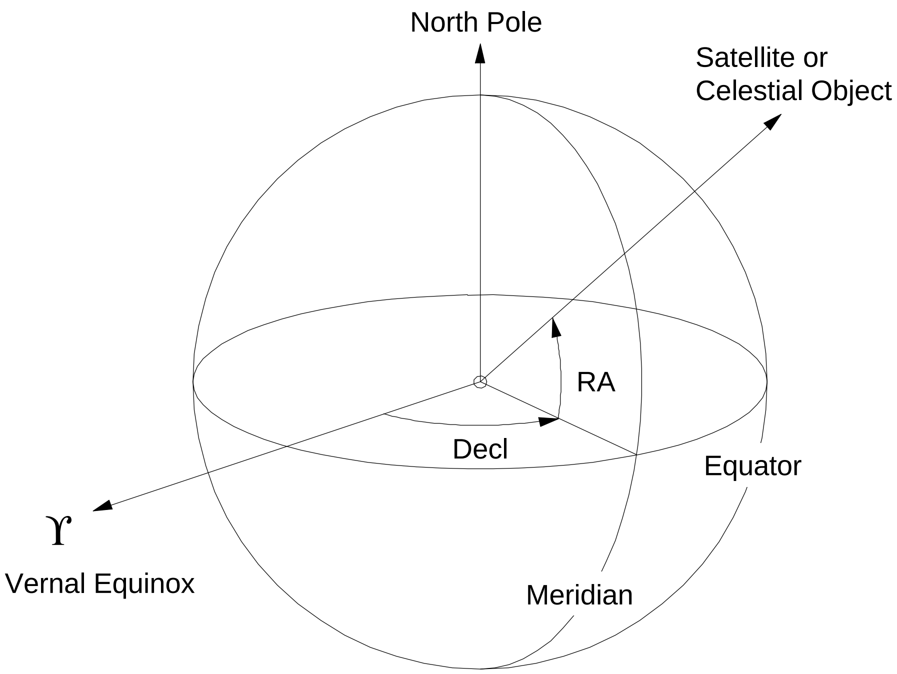


Figure 3-2 Right Ascension and Declination

3.10 Delta-v

Speed change needed for a particular change in *orbit parameters*. The direction and size of the delta-v determines which orbit parameters are most affected, and by how much. For instance, a delta-v, orthogonal to the orbit plane at the time of *ascending node* or *descending node* crossing, results in an inclination change (this is a manoeuvre which requires a relatively large amount of propellant. Tight inclination control therefore limits the lifetime of a satellite in orbit considerably).

3.11 Descending Node

Point in the equatorial plane where the satellite crosses through the equatorial plane in a southerly direction (see Figure 3-1 Satellite Orbit in Space).

3.12 Drift Orbit

A new geostationary satellite is usually delivered in an orbit, which is slightly higher or lower than its final orbit. It then appears to drift slowly towards its final location. The satellite may be halted temporarily (using a *Hohmann Transfer*) at a different location to allow it to be tested without causing interference, after which it is drifted again to its final location.

3.13 Eccentric Anomaly (E)

Auxilliary angle which is used in the integration of Newton's equations for elliptical motion. E is the angle between the main axis and the line from the centre of the ellipse to a point Q on the circle which has been circumscribed about the ellipse. The point Q is a projection of the satellite along a line which is parallel to the minor axis of the ellipse.

The angle E appears in the famous *Kepler's Equation* which relates E, M, and i.

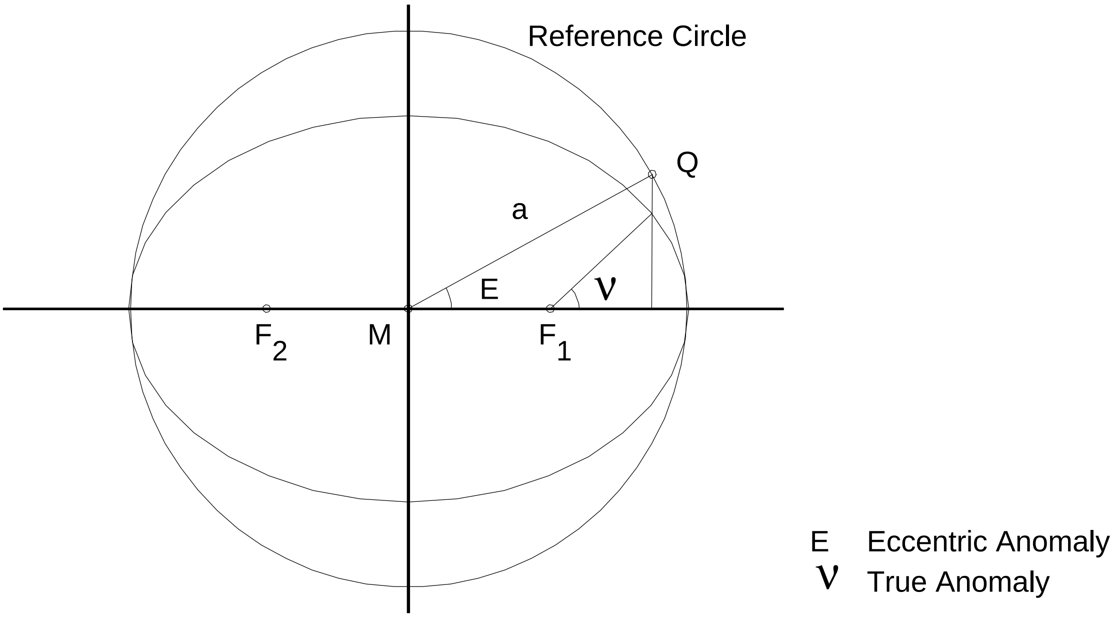
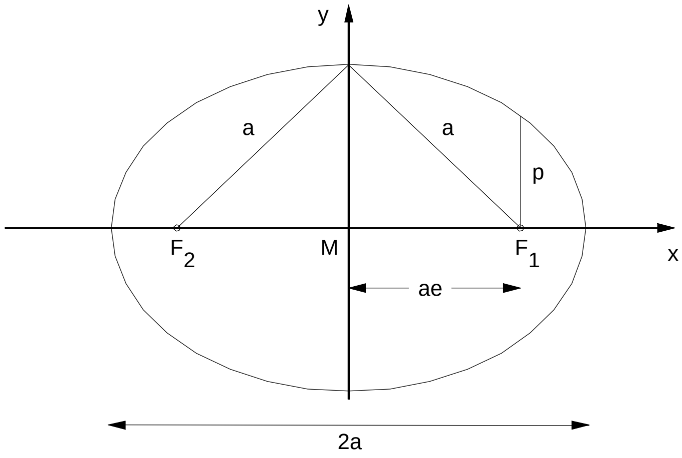


Figure 3-3 True and Eccentric Anomaly

3.14 Eccentricity (e)

Constant defining the shape of the orbit. For an elliptical orbit: e = c/a, where c is the distance from a focal point to the centre of the ellipse, and a is the *semi-major* axis;

$e = 0$	circular orbit
$0 < e < 1$	elliptical orbit
$e = 1$	parabolic orbit
$e > 1$	hyperbolic orbit



a: semimajor axis
b: semiminor axis
e: eccentricity, $e^2 = \frac{\left(a^2 - b^2\right)}{a^2}$
p: semi-latus rectum, $p = a\left(1 - e^2\right)$
F1,F2: Focal Points

Figure 3-4 Some basic properties of the ellipse

3.15 Eclipse

Passage of a satellite through the Earth's shadow. For a geostationary satellite there are two periods in the year of about 40 days duration, centred around vernal and autumnal *equinox*, when a satellite passes through the shadow of the Earth once every day. A communications satellite needs to be equipped with batteries in order to avoid interruptions of traffic during eclipse.

3.16 Ecliptic

The plane of the earth's revolution around the sun.

3.17 Ephemeris Time (ET)

Measurements with highly accurate atomic clocks show that the rotation period of the Earth is slightly irregular. Ephemeris Time is introduced to remove the dependence on the Earth rotation, and is calculated from the observed motion of the Moon. In practice, differences between the rates of ET and *Universal Time* (UT) may be neglected. The absolute difference has increased over the last 100 years to about 60 seconds.

3.18 Epoch

Date and time chosen as the reference date/time from which time is measured. A set of orbital elements is valid for a specified epoch.

3.19 Equation of the Centre

Relation between *True and Mean Anomaly*, used as a first approximation to *Kepler's Equation*. In its simplest form: $\nu = M + 2e \sin M$, where ν =*True Anomaly*, M = *Mean Anomaly*, and e = *eccentricity*.

3.20 Equation of Time

Difference between the *Mean Solar Time* and the real solar time. This difference varies between a minimum of -15 and a maximum of +15 minutes during the year.

3.21 Equinox

Moment at which the Sun as viewed from the Earth appears to cross the celestial equator. This occurs at about 21 March - the vernal equinox - and at about 22 September - the autumnal equinox.

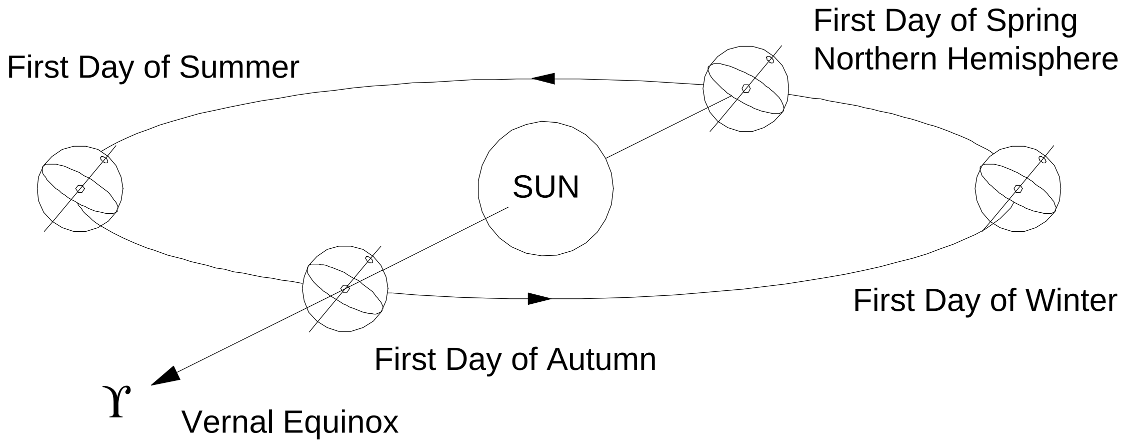


Figure 3-5. Vernal Equinox direction and seasons (Northern Hemisphere)

3.22 Figure of the Earth

The shape of the Earth can be approximated by an ellipsoid. This spheroid of revolution, is a geometrical shape in which any cross-section parallel to the equator is a circle, and any cross-section through the north-south axis is an ellipse of which the minor axis coincides with the Earth's axis. One of the many reference ellipsoids in use is WGS 1984, with the following parameters:

Equatorial Radius (a) =	6378 137.0 m
Polar Radius (b) =	6356 752.3 m
Flattening (f) =	1/298.257223563

3.23 Footpoint of a Satellite

Point on the surface of the Earth, directly below a satellite. The footpoint is the intersection of the Earth's surface and the line connecting the centre of the Earth and the satellite.

3.24 Footprint

That portion of the earth's surface from which the elevation angle towards a satellite exceeds a specified value (usually 0, 5, or 10 degrees), see Figure 3-6 Footpoint and Footprint of a satellite.

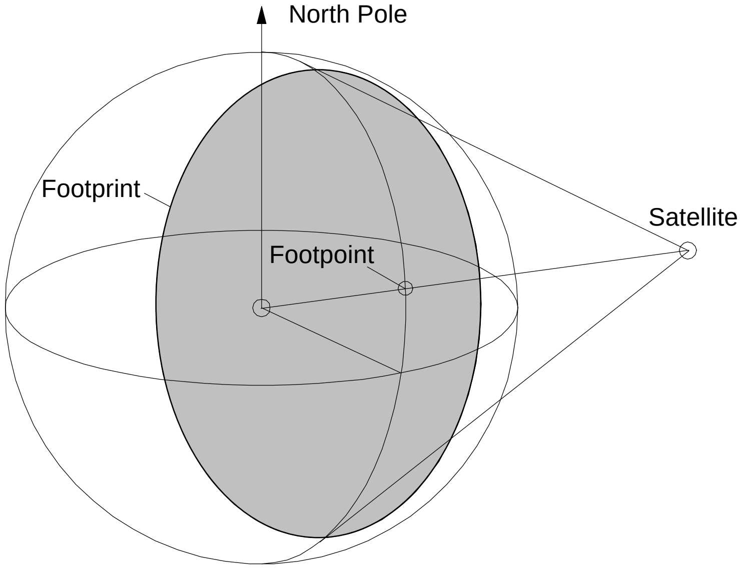


Figure 3-6 Footpoint and Footprint of a satellite

3.25 Free-Space Loss

Signal attenuation that would occur on a link between an isotropic antenna on the surface of the earth and an isotropic antenna onboard a satellite in the absence of any propagation effects such as atmospheric absorption, diffraction and obstruction.

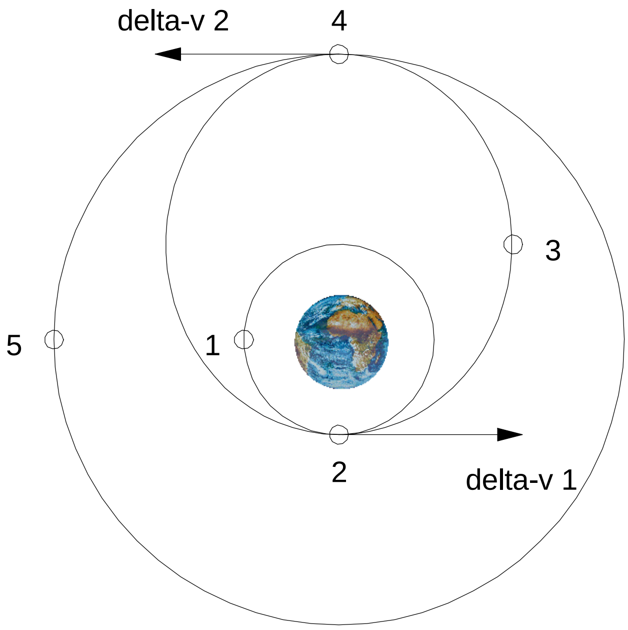
3.26 Geostationary Orbit

Circular *prograde* orbit in the equatorial plane of the earth with an orbital <>period of exactly one *sidereal day*. The radius of a geostationary orbit is 6.6107(equatorial) earth radii.

3.27 Greenwich Mean Time (GMT)

Mean Solar Time at the meridian of Greenwich, England, formerly used as a basis for standard time. GMT is replaced by *Universal Time* (UT).

3.28 Hohmann Transfer



Transfer Sequence:

1: satellite in low circular orbit
2: first delta-v
3: satellite in elliptical orbit
4: second delta-v
5: satellite in high circular orbit

Figure 3-7. Hohmann Transfer

3.29 Hour Angle

Angle measured westward from the observer's meridian to the meridian that contains the direction to a celestial object.

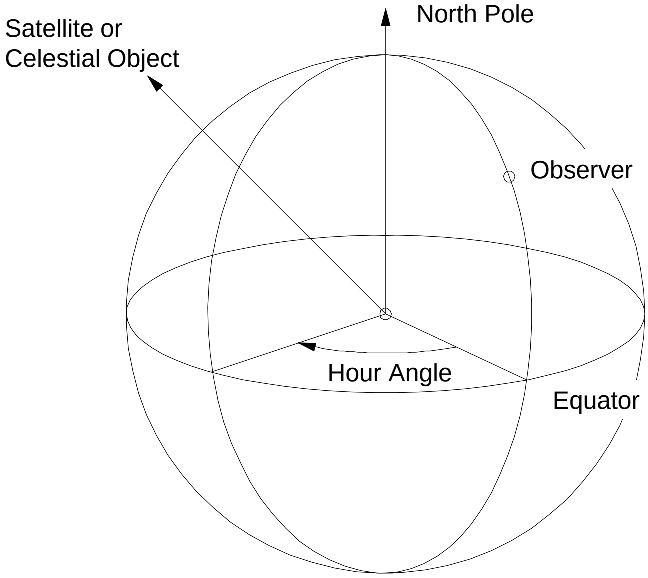


Figure 3-8. Hour Angle

3.30 Inclination of an Orbit (i)

The angle between the plane of the Earth's equator and the plane containing the orbit (counted positively in the direction towards the North Pole), see Figure 3-1 Satellite Orbit in Space

3.31 Inclined Orbit

Any non-equatorial orbit of a satellite (see inclination).

3.32 Julian Date (JD)

The sequential day count reckoned consecutively beginning on 1 January 4713 BC (The Julian Date for 1 January 2000 was 2,451,545.0).

3.33 Keplerian Elements

A set of six parameters, which together describe shape and orientation of an elliptical orbit around the earth, as well as the position of a satellite in that orbit at a given *epoch*. The usual elements are: *Right Ascension* of the *Ascending Node, Argument of the Perigee*, *Mean Anomaly*, *Semi-Major Axis*, *Inclination* and *Eccentricity*.

3.34 Kepler's Equation

Equation which relates *Mean Anomaly* (M), *Eccentric Anomaly* (E) and *Eccentricity* (e). This equation, $M = E - e * \sin(E)$, is used to calculate E for given M and eccentricity e. It is possible to calculate the *True Anomaly* from E.

3.35 Line of Nodes

Line of intersection of the equator and the orbital plane. This line goes through the *ascending and descending nodes*, see Figure 3-1 Satellite Orbit in Space

3.36 Major Axis of an ellipse

The longest diameter of the ellipse, which goes through the centre and both focal points, is called the major axis, see Figure 3-4 Some basic properties of the ellipse.

3.37 Mean Anomaly (M)

Angle, measured from the *periapsis* in the direction of the satellite's motion, which a satellite would sweep out if it moved at a constant angular speed, i.e. M =2*pi*t/T (radians), where T is the orbital *Period*.

3.38 Mean Motion (n)

The average angular velocity of a satellite in an elliptical orbit, i.e. n=2*pi/T (radians/second), where T is the orbital *Period*.

3.39 Mean Solar Time or Universal Time (UT)

Time measured with respect to the motion of a fictitious body called the Mean Sun, which moves at a constant rate (another way to state this assumption is that the earth moves in a circular orbit around the sun, and that the axis of rotation is perpendicular to the orbital plane (ecliptic). The time interval between two meridian crossings of the Mean Sun is exactly one solar day. Due to the combined effects of the eccentricity of the Earth's orbit and the tilt of the Earth rotation axis, the real sun arrives at your local meridian a little early at certain times of the year, and a little late at other times. The difference between real solar time and mean solar time is called the *Equation of Time*

Measurements with highly accurate atomic clocks show that the rotation period of the Earth is slightly irregular. *Ephemeris Time* (ET), introduced to remove the dependence on the Earth rotation, is calculated from the observed motion of the Moon. In practice, differences between the rates of ET and UT may be neglected. The absolute difference has increased over the last 100 years to about 60 seconds.

3.40 Node Rotation

The non-spherical shape (oblateness) of the Earth causes a rotation of the orbital plane. This precessional motion is similar to that of a simple top: the normal to the orbital plane sweeps out a cone shaped surface in space with a semi-vertex angle equal to the inclination i. As the orbit precesses, the line of intersection of the equator and the orbital plane (the *line of nodes*) rotates westward for a *prograde* orbit and eastwards for a *retrograde* orbit. This effect is known as node rotation.

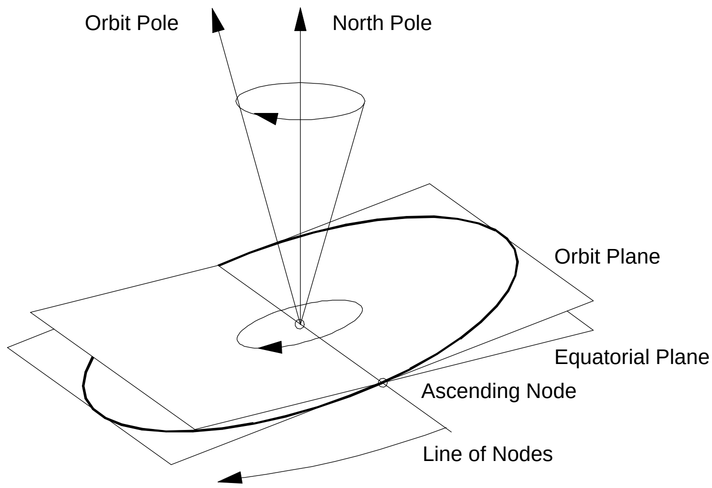


Figure 3-9 Precession and Node Rotation

3.41 Orbit Parameters

The shape and orientation of an elliptical orbit around the earth, and the position of a satellite in that orbit at a given epoch, can be described with a set of six parameters (also referred to as *Keplerian elements*). The usual elements are: *Right Ascension* of the *Ascending Node*, *Argument of the Perigee*, *Mean Anomaly*, *Semi-Major Axis*, *Inclination* and *Eccentricity*.

3.42 Osculating Orbit

Orbit along which a satellite would move if all perturbing accelerations were removed at a particular time. At that time, or Epoch, the osculating and true orbits are in contact. *(Orbit parameters* are always given for the *osculating orbit*, because the true, perturbed, orbit cannot be described this way).

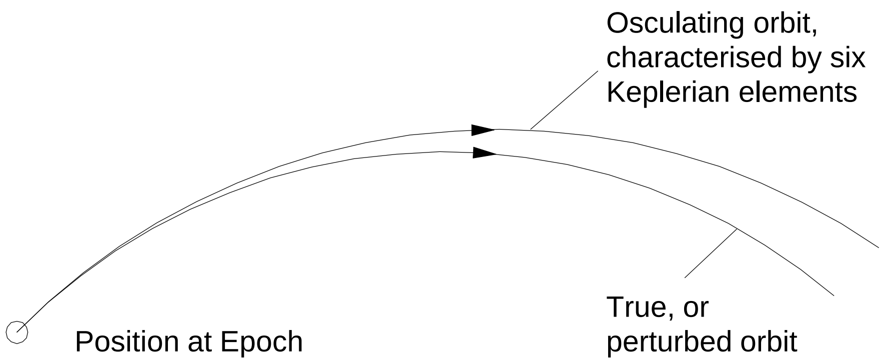


Figure 3-10 Osculating Orbit

3.43 Periapsis

The point of an elliptical orbit that is closest to the gravitational centre of the system consisting of the primary body and the satellite (In Earth-based systems, the periapsis is called *perigee*).

3.44 Perigee

The point in the orbit of a satellite, orbiting the earth, which is closest to the gravitational centre of the earth (See Figure 3-1 Satellite Orbit in Space).

3.45 Perigee Altitude or Height

The altitude of Perigee above a specified reference point serving to represent the surface of the earth.

3.46 Period (T)

Time for a satellite to complete one revolution around the centre of gravity.

3.47 Perturbation

Deviation from true elliptical motion of a satellite caused by disturbing accelerations due to the non-spherical shape of the Earth, influence of sun and moon, drag and sun radiation pressure.

3.48 Precession

Rotation of the orbital plane caused by the non-spherical shape (oblateness) of the Earth. Precessional motion is similar to that of a simple top: the normal to the equatorial plane sweeps out a cone shaped surface in space with a semi-vertex angle equal to the inclination i. As the orbit precesses, the line of intersection of the equator and the orbital plane (the *line of nodes*) rotates westward for a *prograde* orbit and eastwards for a *retrograde* orbit (see Figure 3-9 Precession and Node Rotation).

3.49 Prograde Orbit

Orbit of a satellite orbiting the earth, in which the projection of the satellite's position on the (Earth's) equatorial plane revolves in the direction of the rotation of the Earth. (the orbit *inclination* of a satellite in a prograde orbit is less than 90 degrees).

3.50 Retrograde Orbit

Orbit of a satellite orbiting the earth, in which the projection of the satellite's position on the (Earth's) equatorial plane revolves in the direction opposite to that of the rotation of the Earth. (the orbit *inclination* of a satellite in a retrograde orbit is in excess of 90 degrees).

3.51 Right Ascension

Angle in the plane of the equator, measured eastward from the *vernal equinox* direction to a particular meridian, e.g. the meridian which contains the vector pointing in the direction of the satellite (see Figure 3-2 Right Ascension and Declination).

3.52 Right Ascension of the Ascending Node (RAAN, or Ω)

Angle in the equatorial plane between the direction to the *vernal equinox* and the direction to the *ascending node*, measured counter-clockwise when viewed from the north side of the equatorial plane (see Figure 3-1 Satellite Orbit in Space).

3.53 Round-trip Delay Time

The time required for a signal to travel from an earth station via a satellite to another earth station (the round -trip delay is approximately 7 ms for a satellite in Low Earth Orbit, 70 ms for a satellite in Medium Earth Orbit, and approximately 250 ms for a geostationary satellite).

3.54 Semi-major Axis of an Ellipse

The longest diameter of the ellipse, which goes through the centre and both focal points, is called the *major axis*. The portion from the centre of the ellipse in either direction is the Semi-major Axis (see Figure 3-4 Some basic properties of the ellipse).

3.55 Sidereal Time

Time required for the Earth to rotate once on its axis relative to the stars. This occurs in 23h 56m 4s of ordinary mean solar time. A sidereal day consists of 24 sidereal hours. The sidereal day starts when the *Vernal Equinox* crosses the Greenwich meridian. Sidereal time is therefore equal to the *Hour Angle* of the *Vernal Equinox*.

3.56 True Anomaly (f, or ν)

The angle in the plane of the satellite orbit between the *periapsis* and the satellite position, measured in the direction of the satellite's motion (see Figure 3-3 True and Eccentric Anomaly).

3.57 Vernal Equinox Direction

Direction towards a point in the constellation of Aries. On the first day of spring, a line joining the centre of the Earth and the centre of the sun points in this direction. This line is the intersection of the earth's equatorial plane and the ecliptic plane, which is the plane of the earth's revolution around the sun. The vernal equinox direction is used as the x-axis for an astronomical reference system (if extreme precision were needed, it would be necessary to specify that the reference frame is based on the vernal equinox for a particular *epoch*) (see Figure 3-5. Vernal Equinox direction and seasons (Northern Hemisphere)).

3.58 Universal Time (UT)

Local mean solar time on the Greenwich meridian, also *called Greenwich Mean Time* (GMT), or Zulu Time (Z).

4 ORBITAL MOTION THEORY

4.1 KEPLER'S LAWS

Kepler's three laws describe the observed motion of the planets around the sun. They are equally valid in describing the motion of an artificial satellite around the earth (ignoring any orbit perturbations for the time being):

First Law The orbit of a satellite is an ellipse with the gravitational centre of the earth at its focus.

Second Law The radius vector from the focus towards the satellite sweeps out equal areas in equal periods of time.

Third Law The square of the orbital period of a satellite is proportional to the cube of its mean distance to the focus.

4.2 NEWTON'S LAWS OF MOTION

Newton published the following three remarkably simple basic laws of motion, which together with his Law of Universal Gravitation form the physical basis for all theoretical work:

First Law Every object continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

Second Law The rate of change of momentum measured relative to an inertial reference frame, is proportional to the force impressed and is in the same direction of that force.

Third Law To every action there is always opposed an equal reaction.

The second law can be expressed mathematically as follows:

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2}, \tag{4.1}$$

where **F** is the vector sum of all forces acting on the mass m and $\frac{d^2\mathbf{r}}{dt^2}$ is the vector acceleration of the mass measured relative to an inertial reference frame.

4.3 NEWTON'S LAW OF UNIVERSAL GRAVITATION

In addition to stating the three laws of motion, Newton formulated the following law of universal gravitation:

> Any two objects attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them.

This law can be expressed mathematically in vector notation as:

$$\mathbf{F} = \frac{Gm_1m_2}{r^3}\mathbf{r} \tag{4.2}$$

where **F** is the force on mass m_1 due to mass m_2. **F** is a vector in the direction from m_1 to m_2. G is the Universal Gravitational Constant.

4.4 PROOF OF KEPLER'S LAWS, OR SOLUTION OF THE TWO-BODY PROBLEM

The two-body problem was first stated and solved by Newton. The importance of this problem lies in two facts: Firstly, the problem involving spherical bodies in which the mass is distributed in spherical shells, is the only gravitational problem that can be solved rigorously (and relatively simply). Secondly, practical problems of orbital motion can be treated as approximate two-body problems, i.e. the two-body solution may be used to provide a first approximation of the orbital motion, and is therefore used as the starting point of the calculation of more accurate solutions.

The mathematical formulation of the two body problem results from a combination of equations (4.1) and (4.2). We choose the centre of the earth as the origin of our co-ordinate system, and we define the positive sense of the radius vector **r**, as the direction away from the origin. We first rewrite expressions (4.1) and (4.2) so that they express the force acting on the satellite with mass m, due to the large mass M of the earth:

$$\mathbf{F}_m = m\frac{d^2\mathbf{r}}{dt^2}$$

$$\mathbf{F}_m = -GmM\frac{\mathbf{r}}{r^3} \quad \text{(a minus sign because the force is towards the origin)}$$

$$\text{or,} \quad \frac{d^2\mathbf{r}}{dt^2} + GM\frac{\mathbf{r}}{r^3} = \frac{d^2}{dt^2}\mathbf{r} + \mu\frac{\mathbf{r}}{r^3} = 0 \tag{4.3}$$

The integration of this basic equation, "The Two-Body Equation of Motion", is relatively straightforward for someone with a basic knowledge of Vector Analysis, and leads to the proof of Kepler's laws.

4.4.1 STEP 1 Proof that h is a constant vector

Cross multiply equation (4.3) by **r**:

$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r}\times\mathbf{r} = 0$. The second term is zero since $\mathbf{r}\times\mathbf{r} = 0$. Hence

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = 0 \tag{4.4}$$

Take the derivative of the angular momentum **h**:

$$\frac{d\mathbf{h}}{dt} = \frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right) = \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt} = 0 \tag{4.5}$$

The first term on the right is zero because of (4.4). The second term is zero because the cross product of two equal vectors is zero. Hence: $\frac{d\mathbf{h}}{dt} = 0$, or in other words: **h is a constant vector.**

4.4.2 STEP 2 Proof of Kepler's first law

.
Cross multiply equation (3) by **h**:

$$\frac{d^2\mathbf{r}}{dt^2}\times\mathbf{h} = -\frac{\mu}{r^3}\mathbf{r}\times\mathbf{h} = -\frac{\mu}{r^3}\mathbf{r}\times\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)$$

Using the expression for the triple vector product, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\bullet\mathbf{c})\mathbf{b} - (\mathbf{a}\bullet\mathbf{b})\mathbf{c}$, the last expression becomes:

$$-\frac{\mu}{r^3}\left[\left(\mathbf{r}\bullet\frac{d\mathbf{r}}{dt}\right)\mathbf{r} - (\mathbf{r}\bullet\mathbf{r})\frac{d\mathbf{r}}{dt}\right] = -\frac{\mu}{r^3}\left(r\frac{dr}{dt}\mathbf{r} - r^2\frac{d\mathbf{r}}{dt}\right) = \mu\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}\right) = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right),$$

Hence:

$$\frac{d^2\mathbf{r}}{dt^2}\times\mathbf{h} = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$ This equation may be integrated directly, since h is constant:

$$\frac{d\mathbf{r}}{dt}\times\mathbf{h} = \mu\left(\frac{\mathbf{r}}{r}\right) + \mathbf{c},$$ where **c** is a constant (vector) of integration. For reasons, which will become clear later, we write **c** as $\mu\mathbf{e}$. The last expression then becomes:

$$\frac{d\mathbf{r}}{dt}\times\mathbf{h} = \frac{\mu}{r}(\mathbf{r} + \mathbf{e}r) \qquad (4.6)$$

Finally, take the dot product of expression (6) and **r**, using $\mathbf{a}\bullet\mathbf{b}\times\mathbf{c} = \mathbf{c}\bullet\mathbf{a}\times\mathbf{b}$:

$$\mathbf{r}\bullet\left(\frac{d\mathbf{r}}{dt}\times\mathbf{h}\right) = \frac{\mu}{r}\left(\mathbf{r}\bullet\mathbf{r} + \mathbf{e}\bullet\mathbf{r}\, r\right)$$, or:

$$\mathbf{h}\bullet\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right) = \mathbf{h}\bullet\mathbf{h} = h^2 = \frac{\mu}{r}\left(r^2 + e\, r^2\cos\nu\right)$$, where ν is the angle between **e** and **r**.

Hence:

$$r = \frac{h^2/\mu}{1 + e\cos\nu} \qquad (4.7)$$

Expression (7) is the general equation in polar co-ordinates for a conic section with the origin at a focal point. If 0 <= e <= 1, the orbit is an ellipse. **This proofs Kepler's first law**.

4.4.3 STEP 3 Proof of Kepler's second law

Rewrite the expression for **h** using $\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$ (see Figure 4-1) (4.8)

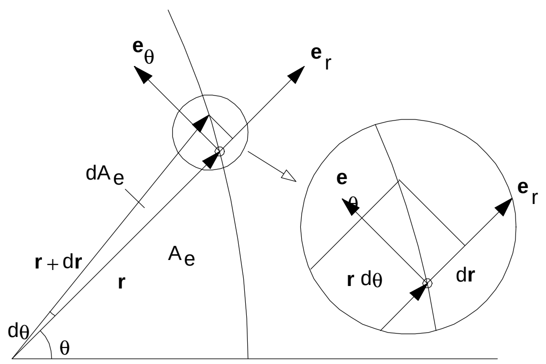


Figure 4-1

$$\mathbf{h} = \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{r} \times \left(\frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta \right) = \frac{dr}{dt}\mathbf{r} \times \mathbf{e}_r + r^2 \frac{d\theta}{dt}\mathbf{e}_r \times \mathbf{e}_\theta \tag{4.9}$$

In the above expression, the first term on the right is zero because **r** and $\mathbf{e}_r$ are collinear vectors.
Expression (4.8) can therefore be rewritten:

$$h = r^2 \frac{d\theta}{dt} \tag{4.10}$$

Inspection of figure 1 shows that $r^2 d\theta = 2dA_e$, (4.11)
i.e. twice the area swept by the radius vector per unit time. Since h is constant, it follows **that the satellite sweeps out equal areas in equal periods of time**, which proofs **Kepler's second law**.

4.4.4 STEP 4 Calculation of the orbital period T, Kepler's third law

The numerator of expression (4.7) is the semi-latus rectum p. Hence:

$$\frac{h^2}{\mu} = p = a\left(1-e^2\right), \text{ or}$$

$$h = \left[\mu a\left(1-e^2\right)\right]^{1/2} \tag{4.12}$$

Combining expressions (4.10), and (4.11), it follows that:

$$h = 2\frac{dA_e}{dt}$$

Since h is constant, it follows that $A_e(t) = \frac{h}{2}t = \frac{\left[\mu a\left(1-e^2\right)\right]^{1/2}}{2}t$ (4.13)

When t = T, or one orbital period, $A_e = \pi ab = \pi a^2\left(1-e^2\right)^{1/2}$ (4.14)

Combining (13) and (14), it follows that:

$$\frac{\left[\mu a\left(1-e^2\right)\right]^{1/2}}{2}T = \pi a^2\left(1-e^2\right)^{1/2} \text{ ,or:}$$

$$T = 2\pi\frac{a^{3/2}}{\mu^{1/2}} \tag{4.15}$$

This proofs Kepler's third law.

4.4.5 TIME DEPENDENCE, MEAN MOTION (n) and MEAN ANOMALY (M)

If a satellite moves through perigee at time t_0, then at time t, the radius vector has swept out an area ΔA_e. In accordance with Kepler's third law:

$\Delta A_e = A_e \frac{t-t_0}{T}$, where $A_e = \pi ab$ is the total area of the ellipse, and T is the orbital period. We now rewrite expression (4.15) for T:

$T = 2\pi\frac{a^{3/2}}{\mu^{1/2}} = \frac{2\pi}{n}$, and we call n the "**Mean Motion**":

$$n = \frac{2\pi}{T} = \frac{\mu^{1/2}}{a^{3/2}} \tag{4.16}$$

Substitute (16) in the expression for ΔA_e, which becomes:

$$\Delta A_e = A_e \frac{n(t - t_0)}{2\pi} = A_e \frac{M}{2\pi}\text{, where } M = n(t - t_0) \qquad (4.17)$$

M is an angle, increasing in time at a constant rate n. We call M the "**Mean Anomaly**". The physical significance of M becomes clear when we consider the case for zero eccentricity, i.e. circular motion. In that case, M describes the position of the satellite at any point in time.

4.4.6 Position In the Orbit. STEP 1. Relation Between Mean And Eccentric Anomaly

The mean anomaly M can be determined for each point in time from expression (4.17). From there to the true anomaly is a two-step process. In Step 1, the eccentric anomaly E is calculated, using an expression, known as **Kepler's Equation**, which relates E to the mean anomaly M. This expression can be derived from a simple geometric consideration. Referring to Figure 4-2 where the point P lies on the circle circumscribed around the ellipse, it follows that:

$$\text{Area MPR} = \pi a^2 \frac{E}{2\pi} = \frac{a^2}{2} E$$

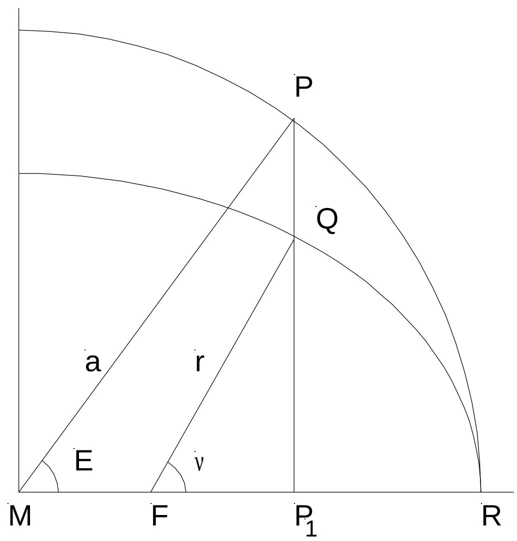


Figure 4-2

$$\text{Area MPP}_1 = \frac{a^2}{2}\cos E \sin E \Rightarrow \text{Area PP}_1\text{R} = \frac{a^2}{2}(E - \cos E \sin E)$$

Using $QP_1 / PP_1 = b/a \Rightarrow \text{Area QP}_1\text{R} = \frac{b}{a}\text{Area PP}_1\text{R} = \frac{ab}{2}(E - \cos E \sin E)$

$$\text{Area FQP}_1 = \frac{1}{2}QP_1 = \frac{1}{2}\left(\frac{b}{a} x a \sin E\right)(a\cos E - ae) = \frac{ab}{2}\sin E(\cos E - e)$$

$$\text{Area FQR} = \text{Area FQP}_1 + \text{Area QP}_1\text{R} = \frac{ab}{2}(E - e\sin E)$$

According to expression (17): $M\frac{ab}{2}$. Therefore, combining the two last expressions:

$$M = E - e\sin E \qquad (4.18)$$

4.4.7 Solution Of Kepler's Equation

E can be derived from expression (18), using a method of successive approximations. For instance, let E_0 = M be a first approximation, and let the true value of E be $E_0 + \Delta E_0$. Then expression (4.18) becomes:

$$M = (E_0 + \Delta E_0) - e\sin(E_0 + \Delta E_0) = E_0 + \Delta E_0 - e\sin E_0 \cos\Delta E_0 - e\cos E_0 \sin\Delta E_0$$

If ΔE_0 is small, this can be approximated as follows:

$M \approx E_0 + \Delta E_0 - e\sin E_0 - e\Delta E_0 \cos E_0$, or

$$\Delta E_0 = \frac{M - E_0 - e\sin E_0}{1 - e\cos E_0} \qquad (4.19)$$

The process can be repeated with $E_0 + \Delta E_0$ as a new approximation to E, until ΔE_0 becomes negligibly small.

4.4.8 Position In The Orbit. STEP 2. Relation Between Eccentric And True Anomaly

Referring again to Figure 4-2, some geometric considerations lead to an expression relating eccentric and true anomaly:

$$QP_1 = \frac{b}{a}PP_1 = \frac{b}{a}a\sin E = b\sin E = (1-e^2)^{1/2}\sin E$$

$$QP_1 = r\sin\nu \Rightarrow \sin\nu = \left(\frac{a}{r}\right)(1-e^2)^{1/2}\sin E \qquad (4.20)$$

$$FP_1 = r\cos\nu = a\cos E - ae \Rightarrow \cos\nu = \left(\frac{a}{r}\right)(\cos E - e) \qquad (4.21)$$

Although expressions (4.20) and (4.21) provide two relations between the eccentric and true anomaly, it is customary to employ a different expression. First express the radius vector r as a function of E:

$$r^2 = FP_1^{\,2} + QP_1^{\,2} = a^2(\cos E - e)^2 + a^2(1-e^2)\sin^2 E$$

After a little reduction, the last expression simplifies to: $r = a(1 - e\cos E)$ (4.22)

Using $\cos^2\frac{a}{2} = \frac{1+\cos a}{2}$, and $\sin^2\frac{a}{2} = \frac{1-\cos a}{2}$, we can rewrite (4.20) and (4.21):

$$\frac{\sin^2\frac{\nu}{2}}{\cos^2\frac{\nu}{2}} = \frac{1-\left(\frac{a}{r}\right)(\cos E - e)}{1+\left(\frac{a}{r}\right)(\cos E - e)} = \frac{r - a(\cos E - e)}{r + a(\cos E - e)} = \frac{a(1-e\cos E) - a(\cos E - e)}{a(1-e\cos E) + a(\cos E - e)} =$$

$$\frac{(1+e)(1-\cos E)}{(1-e)(1+\cos E)} = \left(\frac{1+e}{1-e}\right)tg^2\frac{E}{2}, \Rightarrow tg\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} tg\frac{E}{2}$$

(4.23)

4.4.9 Expression For the Eccentricity Vector (E)

Take the dot product of expression (4.6) and **h**:

$$\mathbf{h} \bullet \left(\frac{d\mathbf{r}}{dt} \times \mathbf{h} \right) = \frac{\mu}{r} (\mathbf{h} \bullet \mathbf{r} + \mathbf{h} \bullet \mathbf{e}\, r)$$

Using $\mathbf{a} \bullet \mathbf{b} \times \mathbf{c} = \mathbf{b} \bullet \mathbf{c} \times \mathbf{a}$, the left-hand side of this expression becomes: $\frac{d\mathbf{r}}{dt} \bullet \mathbf{h} \times \mathbf{h} = 0$ because the cross product of two equal vectors is zero. The first term on the right is also zero because **h** and **r** are perpendicular. Hence $\mathbf{h} \bullet \mathbf{e} = 0$, which means that **the vector e lies in the orbital plane**. Inspection of expression (4.7) shows that r has a minimum when **e** and **r** are collinear. In other words, the vector **e** points in the direction of the perigee, and the angle ν is the **True Anomaly**.

Expression (4.6) can be further developed as follows:

$$\frac{d\mathbf{r}}{dt} \times \mathbf{h} = \frac{\mu}{r} (\mathbf{r} + \mathbf{e} r), \text{ or}$$

$$\frac{d\mathbf{r}}{dt} \times \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt} \right) = \left(\frac{d\mathbf{r}}{dt} \bullet \frac{d\mathbf{r}}{dt} \right) \mathbf{r} - \left(\frac{d\mathbf{r}}{dt} \bullet \mathbf{r} \right) \frac{d\mathbf{r}}{dt} = v^2 \mathbf{r} - (\mathbf{v} \bullet \mathbf{r}) \mathbf{v} = \frac{\mu}{r} (\mathbf{r} + \mathbf{e} r), \text{ or}$$

$$\mathbf{e} = \frac{1}{\mu} \left[\left(v^2 - \frac{\mu}{r} \right) \mathbf{r} - (\mathbf{v} \bullet \mathbf{r}) \mathbf{v} \right] \qquad (4.24)$$

4.4.10 Total Energy Per Unit Mass For Elliptic Motion

The gravitational force is a **conservative force**, which means that it can be described as the gradient of a scalar function V(r):

$$\frac{\mathbf{F}}{m} = -\nabla V(r) = \frac{dV(r)}{dr} \mathbf{e}_r \text{, where} \qquad (4.25)$$

$$V(r) = -\frac{\mu}{r} + C$$

Constant C can be given any convenient value. It is customary to make C = 0, so that $V(\infty) = 0$.

V(r) is called the **potential energy** at point r. It is the work W done by the gravitational force when an object of unit mass moves from its position r to infinity:

$$W = \int_r^\infty \frac{\mathbf{F}}{m} \bullet \mathbf{dl} = \int_r^\infty -\frac{dV(r)}{dr} \mathbf{e}_r \bullet \mathbf{dl} = -\int_r^\infty \frac{dV(r)}{dr} dr = V(r) - V(\infty) = V(r) = -\frac{\mu}{r} \qquad (4.26)$$

Similarly, when the object moves from $\mathbf{r}_1$ to $\mathbf{r}_2$, the work done per unit mass is:

$$W = \int_{r_1}^{r_2} \frac{\mathbf{F}}{m} \bullet \mathbf{dl} = V_1 - V_2 \qquad (4.27)$$

When the object moves from $\mathbf{r}_1$ to $\mathbf{r}_2$, its kinetic energy T per unit mass also changes:

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{\mathbf{F}}{m} \bullet \mathbf{dl} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{d\mathbf{v}}{dt} \bullet \mathbf{v}dt = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{v} \bullet d\mathbf{v} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} d\frac{\mathbf{v} \bullet \mathbf{v}}{2} = T_2 - T_1 \qquad (4.28)$$

Combining (4.27) and (4.28):

$V_1 - V_2 = T_2 - T_1$, or $V_1 + T_1 = V_2 + T_2$.

Thus for an object moving under conservative forces, the total energy per unit mass V + T is constant. Introduce a new constant $\mathcal{E}$, which is the **total energy per unit mass**, and we can write:

$$\mathcal{E} = T + V \qquad (4.29)$$

This is the theorem of **conservation of mechanical energy**. For an object moving in a gravitational field, using (4.26) and (4.28), we can write:

$$\mathcal{E} = \frac{\mathbf{v} \bullet \mathbf{v}}{2} - \frac{\mu}{r} \qquad (4.30)$$

Evaluation of $\mathcal{E}$ at perigee leads to:

$$\mathcal{E} = -\frac{\mu}{2a} \qquad (4.31)$$

Use the following expressions to prove this:

$$h = r_p v_p$$

$$\frac{h^2}{\mu} = p = a\left(1 - e^2\right)$$

$$r_p = a(1 - e)$$

5 SUMMARY OF EXPRESSIONS

5.1 Total Energy per unit mass (ε)

$\varepsilon = T + V$, where T is the total kinetic energy and V is the total potential energy. For an unperturbed orbit, the total energy does not change as the satellite moves along the orbit. However, in a non-circular orbit, there is a constant exchange between potential and kinetic energy. Useful expressions for ε are:

$$\varepsilon = \frac{v \bullet v}{2} - \frac{\mu}{r} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \qquad (5.1)$$

5.2 Angular Momentum (h)

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = rv\sin\gamma = rv\cos\beta \qquad (5.1)$$

The vector **h** is perpendicular to the orbital plane (see also section 5.8 The orbit in space). The angle β between the velocity vector and the local horizon is called the "**flight-path angle**" (**local vertical** is the line through satellite and the centre of the earth. The "**local horizon**" is the line perpendicular to the local vertical.

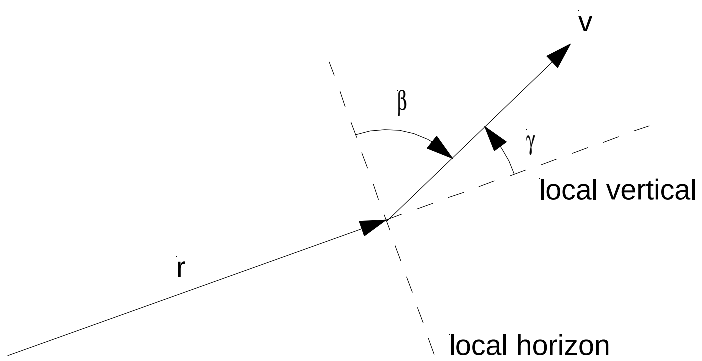


Figure 5-1

For an unperturbed orbit, h is a constant, related to the orbit geometry:

$$\frac{h^2}{\mu} = p = a\left(1 - e^2\right) \qquad (5.2)$$

At perigee and apogee, the flight angle is zero. Hence: $h = r_p v_p = r_a v_a$ (5.3)

5.3 Eccentricity vector (e)

$$\mathbf{e} = \frac{\mathbf{v}\times\mathbf{h}}{\mu} - \frac{\mathbf{r}}{r} = \frac{1}{\mu}\left[\left(\mathbf{v}\bullet\mathbf{v} - \frac{\mu}{\mathbf{r}}\right)\mathbf{r} - (\mathbf{r}\bullet\mathbf{v})\mathbf{v}\right] \qquad (5.4)$$

The vector **e** lies in the plane of the orbit, pointing from the centre of the earth toward perigee (see also section 5.8 The orbit in space).

5.4 Expressions for the radius vector r

$$r = \frac{p}{1 + e\cos\nu} = a(1 - e\cos E) \qquad (5.5)$$

$$r_a = a(1+e), \qquad (5.6)$$

$$r_p = a(1-e), \qquad (5.7)$$

or $r_p + r_a = 2a$ (5.8)

$$e = \frac{r_a - r_p}{r_a + r_p} \qquad (5.9)$$

5.5 (T) and Mean Motion (n)

$$T = \frac{2\pi}{\mu^{1/2}} a^{3/2} \qquad (5.10)$$

$$n = \frac{\mu^{1/2}}{a^{3/2}} \qquad (5.11)$$

5.6 Components of the velocity vector v

Two co-ordinate systems are in use: the r, ν system, with axes along the local vertical and along the local horizon, and the ξ, η system, with axes parallel to the major and minor axes of the orbit (see Figure 5-2. Note: In this figure, both v_ξ and v_r will be positive, whereas v_η and v_ν are negative).

$$v_r = \sqrt{\frac{\mu}{r}\left[2 - \left(\frac{r}{a}\right) - \left(\frac{a}{r}\right)(1-e^2)\right]} = e\sqrt{\frac{\mu}{p}}\sin\nu \qquad (5.12)$$

$$v_\nu = \frac{h}{r} = \sqrt{\frac{\mu}{p}}(1 + e\cos\nu); \qquad (5.13)$$

$$v_\xi = -\sqrt{\frac{\mu}{p}}\sin\nu; \qquad (5.14)$$

$$v_\eta = \sqrt{\frac{\mu}{p}}(e + \cos\nu) \qquad (5.15)$$

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} \qquad (5.16)$$

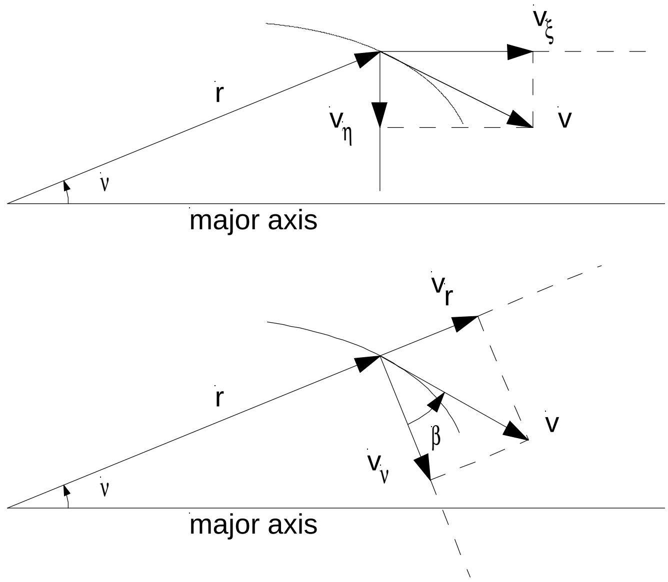


Figure 5-2

At perigee and apogee the velocity vector **v** is perpendicular to the radius vector **r**, and:

$$v_a = \frac{2\pi a}{T}\sqrt{\frac{1-e}{1+e}} \tag{5.17}$$

$$v_p = \frac{2\pi a}{T}\sqrt{\frac{1+e}{1-e}} \tag{5.18}$$

The flight path angle β is given by:

$$\sin\beta = \frac{e\sin\nu}{\sqrt{1+2e\cos\nu+e^2}} \tag{5.19}$$

$$\cos\beta = \frac{1+e\cos\nu}{\sqrt{1+2e\cos\nu+e^2}} \tag{5.20}$$

5.7 Relationship between Mean Anomaly, Eccentric Anomaly and True Anomaly

For an unperturbed orbit $M(t+\tau) = M(t) + n\tau$ (5.21)

$$M = E - e\sin E \quad (5.22)$$

$$\sin\nu = \frac{a\sqrt{(1-e^2)}}{r}\sin E = \frac{\sqrt{(1-e^2)}}{1-e\cos E}\sin E \quad (5.23)$$

$$\cos\nu = \frac{e-\cos E}{e\cos E - 1} \quad (5.24)$$

$$\tan\nu = \frac{\sqrt{(1-e^2)}\sin E}{\cos E - e} \quad (5.25)$$

$$\cos E = \frac{e+\cos\nu}{1+e\cos\nu}; \quad (5.26)$$

$$\sin E = \frac{r\sin\nu}{a\sqrt{(1-e^2)}} \quad (5.27)$$

$$\tan\frac{E}{2} = \left(\frac{1-e}{1+e}\right)^{1/2}\tan\frac{\nu}{2} \quad (5.28)$$

(Note: E and ν are always in the same quadrant)

5.8 The orbit in space

The geocentric-equatorial co-ordinate system, with origin at the earth's centre, is used to describe the position of the satellite in space (see Figure 5-3).

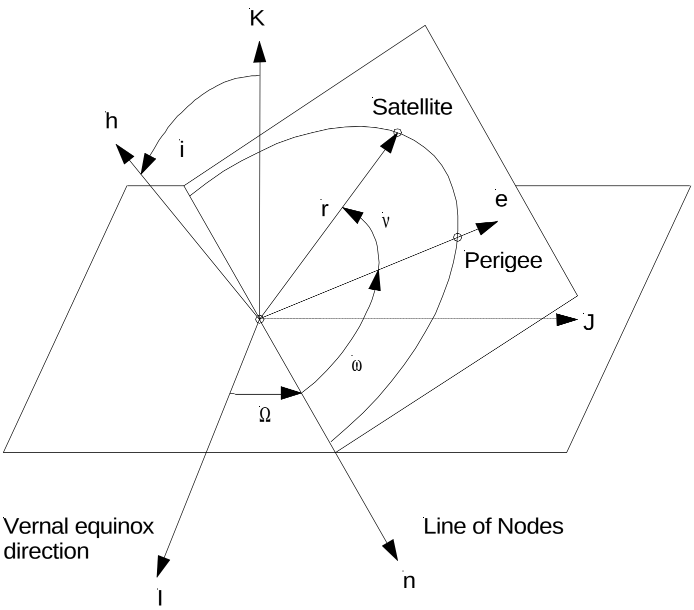


Figure 5-3

The x-axis lies in the equatorial plane and points in the direction of the vernal equinox. **I** is a unit vector along the x-axis. The y-axis lies also in the equatorial plane, and **J** is a unit vector along the y-axis. **K** is the unit vector perpendicular to the equatorial plane. Figure 5-3 shows the angular momentum **h**, the eccentricity vector **e**, and the “node vector” **n**.

$$\mathbf{n} = \mathbf{K} \times \mathbf{h}; \tag{5.29}$$

$$\cos i = \frac{\mathbf{h} \bullet \mathbf{K}}{h} \tag{5.30}$$

$$\cos \Omega = \frac{\mathbf{n} \bullet \mathbf{I}}{n} \tag{5.31}$$

$$\cos \omega = \frac{\mathbf{n} \bullet \mathbf{e}}{ne} \tag{5.32}$$

$$\cos \nu = \frac{\mathbf{e} \bullet \mathbf{r}}{er} \tag{5.33}$$

The co-ordinates x_S, y_S, z_S of the radius vector **r** to the satellite are given by:

$$x_S = r[\cos\Omega\cos(\omega+\nu) - \sin\Omega\sin(\omega+\nu)\cos i] \tag{5.34}$$

$$y_S = r[\sin\Omega\cos(\omega+\nu) + \cos\Omega\sin(\omega+\nu)\cos i] \tag{5.35}$$

$$z_S = r\sin(\omega+\nu)\sin i. \tag{5.36}$$

5.9 Azimuth and Elevation to a Satellite

Use Figure 5-4, which shows the cross section of the (highly exaggerated) elliptical earth and the observer's meridian. Also shown in the figure is an orthogonal co-ordinate system at the observer's location P. This local co-ordinate system has unit vectors $\mathbf{e}_1$ (in the meridian plane, pointing towards local south), $\mathbf{e}_2$ (pointing to local east), and $\mathbf{e}_3$, (pointing to zenith).

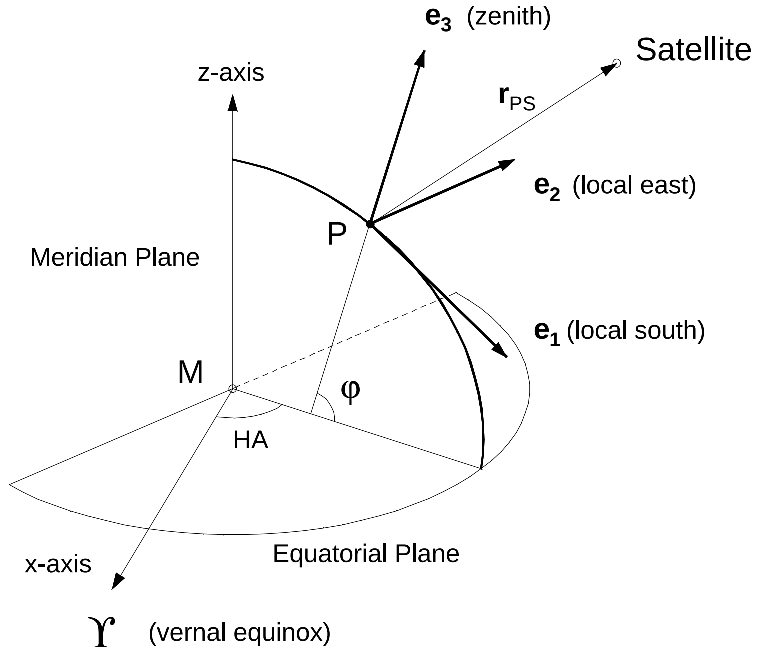
The co-ordinates x_P, y_P, z_P of an observer on the earth surface (using an ellipsoid earth model) are:

$$x_P = \left[\frac{A_E}{\sqrt{1-e^2\sin^2\varphi}} + H\right]\cos\varphi\cos(\lambda+\theta_g) \tag{5.37}$$

$$y_P = \left[\frac{A_E}{\sqrt{1-e^2\sin^2\varphi}} + H\right]\cos\varphi\sin(\lambda+\theta_g) \tag{5.38}$$

$$z_P = \left[\frac{A_E}{\sqrt{1-e^2\sin^2\varphi}} + H\right]\sin\varphi \tag{5.39}$$

where A_E = equatorial radius; e = earth eccentricity
φ = geodetic latitude; H = height above the ellipsoid
l = geodetic longitude HA = $\theta_g + \lambda$ = Hour Angle of the observer
θ_g = Greenwich siderial time (expressed in radians)



$$\mathbf{e}_1 = \begin{pmatrix} \sin\varphi\cos HA \\ \sin\varphi\sin HA \\ -\cos\varphi \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{pmatrix}$$

$$\mathbf{e}_2 = \begin{pmatrix} -\sin HA \\ \cos HA \\ 0 \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{pmatrix}$$

$$\mathbf{e}_3 = \begin{pmatrix} \cos\varphi\cos HA \\ \cos\varphi\sin HA \\ \sin\varphi \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{pmatrix}$$

Figure 5-4

Let $\mathbf{r}_{ps} = (x_s - x_p)\mathbf{I} + (y_s - y_p)\mathbf{J} + (z_s - z_p)\mathbf{K}$ (5.40)

and

$$|\mathbf{r}_{ps}| = \sqrt{(x_s - x_p)^2 + (y_s - y_p)^2 + (z_s - z_p)^2},$$

Then:

$$a_1 = (\mathbf{r}_{ps} \bullet \mathbf{e}_1)/|\mathbf{r}_{ps}| = [(x_S - x_P)\sin\varphi\cos HA + (y_S - y_P)\sin\varphi\sin HA - (z_S - z_P)\cos\varphi]/|\mathbf{r}_{ps}|$$

$$a_2 = (\mathbf{r}_{ps} \bullet \mathbf{e}_2)/|\mathbf{r}_{ps}| = [-(x_S - x_P)\sin HA + (y_S - y_P)\cos HA]/|\mathbf{r}_{ps}|$$

$$a_3 = (\mathbf{r}_{ps} \bullet \mathbf{e}_3)/|\mathbf{r}_{ps}| = [(x_S - x_P)\cos\varphi\cos HA + (y_S - y_P)\cos\varphi\sin HA + (z_S - z_P)\sin\varphi]/|\mathbf{r}_{ps}|$$

Azimuth φ_{AZ}, and elevation φ_{EL} are now given by the following expressions:

$$\varphi_{EL} = \sin^{-1} a_3 \quad (5.41)$$

$$\varphi_{AZ} = -\tan^{-1} a_2/a_1 \quad \text{if } a_1 < 0, \text{ and } a_2 > 0 \quad (5.42)$$

$$\varphi_{AZ} = \pi - \tan^{-1} a_2/a_1 \quad \text{if } a_1 > 0 \quad (5.43)$$

$$\varphi_{AZ} = 2\pi - \tan^{-1} a_2/a_1 \quad \text{if } a_1 < 0, \text{ and } a_2 < 0 \quad (5.44)$$

5.10 Atmospheric Refraction

Radio rays passing through the atmosphere are bent upwards due to the variation of the refractive index with height. The observed elevation towards a satellite exceeds therefore the calculated value by an amount $\Delta\varphi_{EL}$. Approximate values for $\Delta\varphi_{EL}$ are given in the table below (ITU-R Rec 834)

Uncorrected Elevation Angle φ_{EL} (degrees)	Average Angular Correction $\Delta\varphi_{EL}$ (degrees)			
	Polar Continental Air	Temperate Continental Air	Temperate Maritime Air	Tropical Maritime Air
0				
1	0.45			0.65
2	0.32	0.36	0.38	0.47
4	0.21	0.25	0.26	0.27
10	0.10	0.11	0.12	0.14

$\Delta\varphi_{EL}$ can also be expressed by the following formula

$$\Delta\varphi_{EL} = (n-1)\, f(\varphi_{EL}), \tag{5.45}$$

where n is the refractive index of the atmosphere at the observer's location, and $f(\varphi_{EL})$ is a function of the (uncorrected) elevation angle.

$$(n-1) = \frac{77.6}{T}\left(P + 4810\,\frac{e_p}{T}\right)10^{-6} \tag{5.46}$$

with:

T Temperature in Kelvin

P Barometric pressure in hPa (unit adopted by WMO, numerically identical to millibar)

e_p Water vapour pressure in hPa.

$$e_p = RH\, e_{s0}\, \exp\left(\frac{l_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right), \tag{5.47}$$

where:

RH Relative humidity

e_{s0} saturation vapour pressure in hPa

l_v latent heat of vapour pressure at 0 degr C (or 273.15 K) = 6.1121 hPa

R_v gas constant for water vapour (461.5 joule Kelvin / kg)

T_0 273.15 K

The barometric pressure P in expression (5.48) varies with height:

$$P = P_0 - k_h h \tag{5.48}$$

where,

P_0 atmospheric pressure at sea level

h height above sea level in m

k_h variation of pressure with height, approximately 1 hPa / m

Finally, an approximate formula for $f(\varphi_{EL})$, valid for elevation angles between 0 and 10 degrees is given below (this is just one of the formulas, found in various publications. Does anyone have a better one?):

$$f(\varphi_{EL}) = \tan\left(\frac{\pi}{2} - \varphi_{EL} - \frac{\pi}{180}\right) - 0.00014\tan^3\left(\frac{\pi}{2} - \varphi_{EL} - \frac{\pi}{180}\right) \tag{5.49}$$

Calculated values for $\Delta\varphi_{EL}$ using the above formulas are given in the table below:

φ_{EL} (uncorrected) (degrees)	0	1	2	3	4	5	6	7	8	9	10
$\Delta\varphi_{EL}$	0.57	0.47	0.33	0.26	0.21	0.17	0.15	0.13	0.12	0.10	0.09

(T = 293.15, P = 1020 hPa, RH = 0.5)

5.11 Determining Orbit Parameters from given v and r.

Using the above expressions, this is very straightforward:

Step 1: Calculate **e** (expression 5.4):

e = |**e**|

Step 2: Use expression 5.1 to calculate **h**, and 5.2 to calculate a:

$$a = \frac{h^2}{\mu\left(1 - e^2\right)}$$

Step 3: Use expression 5.30 to calculate i:

$$i = \cos^{-1}\frac{h_K}{h}$$

(Note that i is always less than 180 degrees)

Step 4: Use expressions 5.29 and 5.31 to calculate Ω:

$$\Omega = \cos^{-1}\frac{n_x}{n} = \cos^{-1}\frac{-h_y}{\sqrt{h_x^2 + h_y^2}}$$

(If $n_y > 0$, i.e. if $h_x > 0$, then $\Omega < \pi$)

Step 5: Use expression 5.32 to calculate ω:

$$\omega = \cos^{-1}\frac{\mathbf{n} \bullet \mathbf{e}}{ne} \text{ (if } e_z > 0 \text{ then } \omega < \pi\text{)}$$

Step 6: Use expression 5.33 to calculate ν:

$$\nu = \cos^{-1}\frac{\mathbf{e} \bullet \mathbf{r}}{er} \text{ (if } \mathbf{r} \bullet \mathbf{v} > 0 \text{ then } \nu < \pi\text{)}$$

6 ORBITAL PERTURBATIONS

6.1 Introduction

Previous sections dealt with the fundamental principles of the motion of two mutually attracting objects, assuming that both bodies had a spherically symmetric mass distribution, and moved through empty space. These restrictions allows both objects to be treated as particles, and it could be shown that the solution of the basic differential equation of motion for the two-body system is that of a conic orbit with the centre of mass of the two-body system at one focus.

The actual orbit of a satellite around the earth deviates slightly in shape from that of an ellipse due to various sources, such as the oblateness of the earth, the attraction of both sun and moon and solar radiation. In the case of low earth orbits, atmospheric drag must also be taken into account. The deviations from the ideal case of purely elliptic motion are called "perturbations". Fortunately, the perturbations are small in most practical cases, and as long as the period of observation is less than one orbital period, the orbit closely resembles that of an ellipse. If the observations extend over a long enough period, one can detect changes in both shape and orientation of the orbital plane. The table below summarises the magnitudes of the various accelerations acting on satellites in low orbit and in geostationary orbit.

Table 1. Accelerations in m/s2 on an Earth Satellite

Source	Altitude 750 km	Altitude 1500 km	Geostationnary Altitude
Earth Gravity	7.85	6.42	0.22
Earth Oblateness (J_2) (J_3) (J_4)	$20x10^{-3}$ $0.06x10^{-3}$ $0.04x10^{-3}$	$14x10^{-3}$ $0.04x10^{-3}$ $0.02x10^{-3}$	$160x10^{-7}$ $0.08x10^{-7}$ 0.01x10-7
Atmospheric Drag (Area/Mass = 10 m^2/kg)	$500x10^{-7}$	-	-
Lunar-Solar Attraction	10^{-6}	10^{-6}	$7x10^{-6}$
Solar Radiation Pressure (Area/Mass = 10 m^2/kg)	$500x10^{-7}$	$500x10^{-7}$	$500x10^{-7}$

6.2 Description of Perturbations in terms of the Disturbing Potential

It has been seen before that the gravitational acceleration can be described as the gradient of a potential function: $\frac{d^2\mathbf{r}}{dt^2} = -\nabla V(r)$, where $V(r) = -\frac{\mu}{r}$

When the motion of a satellite can no longer be described sufficiently accurately in terms of two-body motion, in most cases, a disturbing potential R(r) can be introduced so that the equations of motion become:

$$\frac{d^2\mathbf{r}}{dt^2} = -\nabla\left[V(r) + R(r)\right] \qquad (6.1)$$

The form of R depends on the particular type of perturbing source. For perturbations caused by the gravitational attraction of Sun or Moon, R(r) can be written as follows (see Figure 6-1):

$$R = -GM_p\left(\frac{1}{r_{sp}} - \frac{xx_P + yy_P + zz_P}{r_P^3}\right) = -\mu_p\left(\frac{1}{r_{sp}} - \frac{r\cos\phi}{r_P^2}\right), \qquad (6.2)$$

where M_p is the mass of the disturbing body.

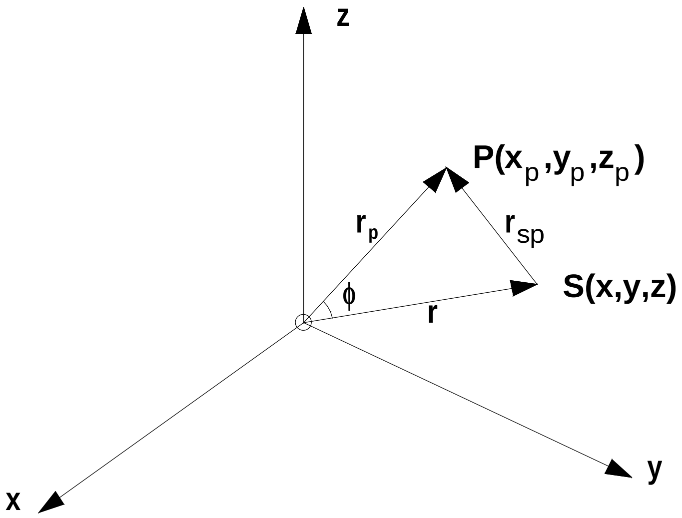


Figure 6-1

The actual shape and mass distribution of the earth can be described in terms of the earth's gravitational potential $U(r,\varphi,\lambda) = V + R$, with the earth centre of mass as origin:

$$U(r,\varphi,\lambda) = -\frac{\mu}{r}\left[1 - \sum_{k=2}^{\infty}\left(\frac{R_e}{r}\right)^k J_k P_k(\sin\varphi) + \sum_{k=2}^{\infty}\sum_{j=1}^{k}\left(\frac{R_e}{r}\right)^k P_k^j(\sin\varphi)\left(C_k^j\cos j\lambda + S_k^j\sin j\lambda\right)\right], \qquad (6.3)$$

where:

R_e: mean equatorial earth radius

φ: latitude

λ: longitude

$P_k(x)$ Legendre functions of degree k defined by: $P_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k$ (6.4)

$P_k^j(x)$: Associated Legendre Polynomials of degree k and order j:

$$P_k^j(x) = (1-x^2)^{j/2}\frac{d^j}{dx^j}P_k(x) \qquad (6.5)$$

The terms with constants J_k and Legendre functions P_k, represent *Zonal* harmonics. These terms are caused by that part of the mass distribution, which is symmetric about the earth's axis, and therefore dependent on latitude only. Lines along which $P_k(\cos\varphi) = 0$, are circles of constant latitude.

The terms containing P_k^j, with j=k, are called *Sectorial* Harmonics. Lines along which $P_k^j(\cos\varphi) = 0$, are meridians. They divide the earth surface in sectors shaped like "orange-slices".

The remaining terms containing P_k^j, are referred to as *Tesserial* Harmonics. The zeros of these functions divide the earth surface into a checkerboard-like array of sectors.
The values of J_k up to degree 7 and C_k^j, S_k^j up to degree 4, as well as the expressions for the corresponding Legendre functions are summarised in Tables 2 and 3 below.

Table 2 Zonal Harmonic Coefficients and Expressions for Pk

J2 = 1082.7 x 10-6	P2(x) = (3x2 - 1) / 2
J3 = -2.56 x 10-6	P3(x) = (5x3 - 3x) / 2
J4 = -1.58 x 10-6	P4(x) = (35x4 - 30x2 +3) / 8
J5 = -0.15 x 10-6	P5(x) = (63x5 - 70x3 + 15x) / 8
J6 = 0.59 x 10-6	P6(x) = (231x6 -315x4 + 105x2 -5) /16
$J_7 = -0.44 \times 10^{-6}$	$P_7(x) = (429x^7 - 693x^5 + 315x^3 - 35x)/16$

Table 3 Sectorial and Tesserial Coefficients and Expressions for Pkj

$C^1_2 = 0$	$S^1_2 = 0$	$P^1_2 = 3x.(1-x^2)^{1/2}$
C22 = 1.57 x 10-6	S22 = -0.897 x 10-6	P22 = 3.(1-x2)
C13 = 2.10 x 10-6	S13 = 0.16 x 10-6	P13 = (1-x2)1/2.(15x2 -3) / 2
C23 = 0.25 x 10-6	S23 = -0.27 x 10-6	P23 = 15x.(1-x2)
C33 = 0.077 x 10-6	S33 = 0.173 x 10-6	P33 = 15.(1-x2)3/2
C14 = -0.58 x 10-6	S14 = -0.46 x 10-6	P14 = (1-x2)1/2.(35x3 - 15x) / 2
C24 = 0.074 x 10-6	S24 = 0.16 x 10-6	P24 = (1-x2).(105x2 - 15) / 2
C34 = 0.053 x 10-6	S34 = 0.004 x 10-6	P34 = 105x.(1-x2)3/2
C44 = -0.0065 x 10-6	S44 = 0.0023 x 10-6	P44 = 105. (1-x2)

6.3 Orbit Propagation Methods

There are two fundamentally different techniques available to describe the effects of perturbing accelerations on a satellite orbit. *Special perturbation techniques* deal with the direct numerical integration of the equation of motion, including all relevant perturbing accelerations. This technique is straightforward and is applicable to any orbit, involving any number of bodies and any perturbing accelerations. However, it has the disadvantage that it does not lead to any generally useable formulae. Furthermore, it requires the accurate calculation of the satellite position at many intermediate points, even if only the position at the end of a given time interval is of interest. Examples of special perturbation techniques are Cowell's method and Encke's method, which are briefly discussed below. *General perturbation techniques* provide an analytical description of the variation with time of the orbit. These techniques are conceptually more complicated. However, they do provide a better physical insight and allow the development of analytical expressions that permit accurate predictions of satellite position to be made over long periods (many orbital periods) with a minimum of computational effort. The method of *Variation of Parameters* falls in this category. The expressions resulting from this method can either be used directly to determine the position of a satellite in space, or they can be used as the basis of further analysis, e.g. to derive expressions for node regression, or for the rotation of the line of absides.

6.4 Cowell's Method

This method was developed by P.H. Cowell and A.D. Crommelin, in the first decade of this century, in order to solve a number of orbit determination problems, including determination of the orbit of the eighth moon of Jupiter, and that of Halley's comet. Cowell's method is a direct numerical integration of the equations of motion, including all perturbations.

The equation of motion for a satellite moving in a perturbed orbit can be written as:

$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r} + \mathbf{a}_p$, where the second term on the right is the perturbing acceleration. This equation is first reduced to two first order equations:

$\frac{d\mathbf{r}}{dt} = \mathbf{v}$, and $\frac{d\mathbf{v}}{dt} = \mathbf{a}_p - \frac{\mu}{r^3}\mathbf{r}$

These equations may be integrated with any suitable numerical method, e.g. Runge-Kutta methods. The obvious advantages of this methods are the simplicity of implementation, and the fact that no prior knowledge of the orbit behaviour is needed. Any number of perturbations and the central force can be handled simultaneously. Its main disadvantage arises from this lack of distinction since a large number of integration steps has to be used due to the large central force term. There is also the danger of loss of accuracy due to accumulation of rounding-off error.

6.5 Encke's Method

Encke's approach requires the use of a reference orbit, the *Osculating Orbit*, along which the satellite would move if at a certain time all perturbing forces would be removed. The orbital elements for the osculating orbit can be used to calculate satellite position and velocity until the actual orbit deviates too far from it. A process of *Rectification* must then be performed to determine a more accurate satellite position, and a set of orbital parameters for a new osculating orbit. This is accomplished by integrating the difference between primary and perturbing accelerations to determine the correction d**r** (see Figure 6-2).

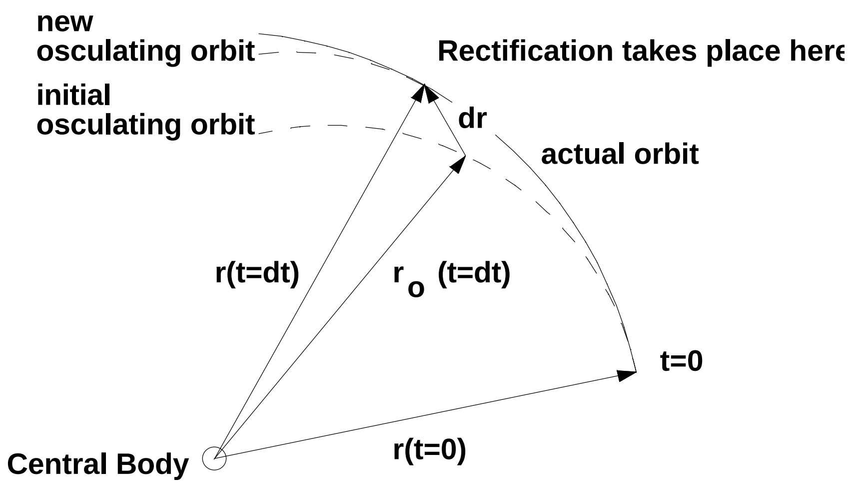


Figure 6-2

In this figure, **r** and $\mathbf{r}_o$ are the radius vectors to true and osculating orbits respectively. A differential equation for the correction $\delta\mathbf{r}$ in the time interval δt can be obtained as follows:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{a}_p$$

$$\frac{d^2\mathbf{r}_o}{dt^2} = -\frac{\mu}{r_o^{\ 3}}\mathbf{r}_o$$

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r}_o \Rightarrow \frac{d^2\delta\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} - \frac{d^2\mathbf{r}_o}{dt^2} = \frac{\mu}{r_o^{\ 3}}\left[\left(1 - \frac{r_o^{\ 3}}{r^3}\right)\mathbf{r} - \delta\mathbf{r}\right] + \mathbf{a}_p$$

This equation can now be integrated numerically over the small time interval δt.

Although developed half a century before Cowell's method, this method can be seen as a refinement of that method, as it requires less computation, and it allows larger integration steps.

6.6 Variation of Parameters Method

This method, first developed by Euler in 1748, and refined by Lagrange in 1783, uses the rate of change of the orbital elements due to the perturbing potential or the perturbing accelerations. Euler showed that it is possible to derive expressions for the rates of change of the orbital elements. If the orbital elements are known at a given point in time, a new set of orbital elements for a later time can then be determined by numerical integration of the expressions for the rates of change. Since in most cases, the rates of change vary slowly compared to position and velocity in an orbit, larger step sizes can be used in this method than in the two previously described methods. In some cases it is possible to integrate the expressions for the rates of change analytically, which makes this method one of General Perturbations.

6.7 Rates of Change of Orbital Elements as a Function of Perturbing Potential R

The following set of equations was first obtained by Lagrange. These equations are one form of a whole family of similar equations, known as *Lagrange's Planetary Equations.* This particular set assumes that we are able to express R as a function of $a, e, M, \Omega, \omega, i$:

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{de}{dt} &= \frac{1-e^2}{na^2e}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} \\
\frac{dM}{dt} &= n - \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\,\sin i}\frac{\partial R}{\partial i} \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cot i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} \\
\frac{di}{dt} &= \frac{1}{na^2\sqrt{1-e^2}}\left[\cot i\frac{\partial R}{\partial \omega} - \csc i\frac{\partial R}{\partial \Omega}\right]
\end{aligned} \qquad (6.6)$$

Lagrange's Planetary Equations may be used in this form in all cases where the perturbing forces can be expressed as the gradient of a scalar function. They are for instance equally valid to determine the effect of the earth's oblateness, as the influence of sun and moon on the satellite orbit. However, they can not be used in the case of atmospheric resistance, which is not a conservative force. Gauss developed an extension to Lagrange's equations, in which three mutually perpendicular components of the disturbing acceleration **S**, **T**, and **W**, are used instead of the perturbing potential. This is shown in Figure 6-3, where $u = \omega + \nu$, and:

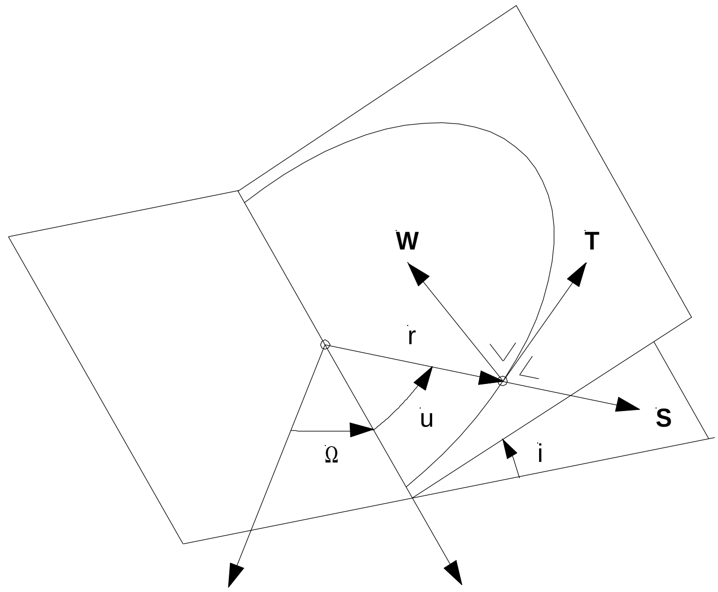


Figure 6-3

$\mathbf{S} = -\frac{\partial R}{\partial r}$ is the component along the radius vector, (6.7a)

$\mathbf{T} = -\frac{1}{r}\frac{\partial R}{\partial u}$ is the component perpendicular to S in the orbital plane, (6.7b)

$\mathbf{W} = -\frac{1}{r \sin u}\frac{\partial R}{\partial i}$ is the component perpendicular to the orbital plane. (6.7c)

The derivatives of R with respect to the orbital elements can be expressed in terms of **S**,**T**, **W**:

$$
\begin{aligned}
\frac{\partial R}{\partial a} &= \frac{r}{a}\mathbf{S} \\
\frac{\partial R}{\partial e} &= -a\cos\nu\,\mathbf{S} + r\sin\nu\left(\frac{1}{1-e^2} + \frac{a}{r}\right)\mathbf{T} \\
\frac{\partial R}{\partial M} &= ae\frac{\sin\nu}{\sqrt{1-e^2}}\mathbf{S} + \frac{a^2}{r}\sqrt{1-e^2}\,\mathbf{T} \\
\frac{\partial R}{\partial \Omega} &= r\cos i\,\mathbf{T} - r\sin i\cos u\,\mathbf{W} \\
\frac{\partial R}{\partial \omega} &= r\mathbf{T} \\
\frac{\partial R}{\partial i} &= r\sin u\,\mathbf{W}
\end{aligned}
\qquad (6.8)
$$

This enables us to write Lagrange's planetary equations in the following useful form:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left(e\sin\nu\,\mathbf{S} + \frac{p}{r}\mathbf{T}\right)$$
$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[e\sin\nu\,\mathbf{S} + \left(\frac{p}{r} - \frac{r}{a}\right)\mathbf{T}\right]$$
$$\frac{dM}{dt} = n + \frac{1}{na^2e}\left[(p\cos\nu - 2re)\mathbf{S} - (p+r)\sin\nu\,\mathbf{T}\right]$$
$$\frac{d\Omega}{dt} = \frac{\sqrt{1-e^2}}{na}\frac{r}{p}\frac{\sin u}{\sin i}\mathbf{W} \qquad (6.9)$$
$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[-\cos\nu\,\mathbf{S} + \sin\nu\left(1+\frac{r}{p}\right)\mathbf{T} - e\left(\frac{r}{p}\right)\cot i\,\sin u\,\mathbf{W}\right]$$
$$\frac{di}{dt} = \frac{\sqrt{1-e^2}}{nae}\left(\frac{r}{p}\right)\cos u\,\mathbf{W}$$

6.8 Accelerations acting on a satellite due to Mass Distribution of the Earth

Combining (6.3) and (6.7) leads to the following useful expressions for the accelerations acting on a satellite due to the mass distribution of the earth (only Zonal terms are included here in order to keep the expressions reasonably compact):

$$S = -\frac{\partial R}{\partial r} = \frac{\mu}{r^2}\sum_{k=2}^{\infty}(k+1)\left(\frac{R_e}{r}\right)^k J_k P_k(\sin u\,\sin i),$$
$$T = -\frac{1}{r}\frac{\partial R}{\partial u} = -\frac{\mu}{r^2}\cos u\,\sin i\sum_{k=2}^{\infty}\left(\frac{R_e}{r}\right)^k J_k\left(\frac{dP_k(x)}{dx}\right)_{x=\sin\varphi}, \qquad (6.10)$$
$$W = -\frac{1}{r\sin u}\frac{\partial R}{\partial i} = -\frac{\mu}{r^2}\cos i\sum_{k=2}^{\infty}\left(\frac{R_e}{r}\right)^k J_k\left(\frac{dP_k(x)}{dx}\right)_{x=\sin\varphi}$$

The derivatives of P_k can easily be found by differentiating the expressions in Table 2a.

6.9 Earth Oblateness Effects: Node Regression and Rotation of the Line of Apsides

An important application of the variation of parameter method is the evaluation of earth oblateness effects on a satellite orbit. The earth oblateness (i.e the ellipsoidal shape) gives rise to the zonal harmonics in expression (6.3), i.e. the terms containing $J_k P_k(\sin\varphi)$. Of these, the one with J_2 is by far the dominant term. For most purposes, a sufficiently accurate estimate of the earth oblateness effects can therefore be found by considering that term only. The disturbing potential R then simplifies to:

$$R = \frac{\mu J_2 R_e{}^2}{2r^3}\left(3\sin^2\varphi - 1\right), \qquad (6.11)$$

and expressions (10) become:

$$S = 3\frac{\mu J_2 R_e^{\ 2}}{r^4}\left(3\sin^2\varphi - 1\right) = 3\frac{\mu J_2 R_e^{\ 2}}{r^4}\left(3\sin^2 u \ \sin^2 i - 1\right)$$

$$T = -\frac{\mu J_2 R_e^{\ 2}}{r^4}\cos u \sin i \frac{\partial}{\partial x}\left(\frac{3x^2 - 1}{2}\right)_{x=\sin\varphi} = -3\frac{\mu J_2 R_e^{\ 2}}{r^4}\sin^2 i \sin u \cos u \qquad (6.12)$$

$$W = -\frac{\mu J_2 R_e^{\ 2}}{r^4}\cos u \frac{\partial}{\partial x}\left(\frac{3x^2 - 1}{2}\right)_{x=\sin\varphi} = -3\frac{\mu J_2 R_e^{\ 2}}{r^4}\sin i \cos i \sin u$$

Substituting (6.12) in Lagrange's planetary equation (6.9) provides the expressions for the rates of change of the orbital elements due to the J_2 term. Analysis of the resulting expressions, and some straightforward but tedious trigonometry, leads to the important conclusions that while all elements are subject to periodic variations, ω, Ω, and M are also changed secularly. The process is shown below for the rate of change of Ω.

Substitution of the expression for T in (6.12) into the expression for $d\Omega / dt$ in (6.9), and using $p = a\left(1 - e^2\right)$, and $n = \mu^{1/2} a^{-3/2}$ leads to:

$$\frac{d\Omega}{dt} = -3\frac{nJ_2 R_e^{\ 2}}{p^2}\left(\frac{a}{r}\right)^3\left(1 - e^2\right)^{3/2}\sin^2 u \cos i$$

The long term average of $\sin^2 u$ is ½. The average of $Q = \left(\frac{a}{r}\right)^3\left(1 - e^2\right)^{3/2}$ can be found by integrating it with respect to M over one orbital period:

$$\overline{Q} = \frac{1}{2\pi}\int_0^{2\pi} Q\,dM = \frac{1}{2\pi}\int_0^{2\pi} Q\left(\frac{r}{a}\right)^2\left(1 - e^2\right)^{-1/2} d\nu = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3\left(1 - e^2\right)^{3/2}\left(\frac{r}{a}\right)^2\left(1 - e^2\right)^{-1/2} d\nu =$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)\left(1 - e^2\right) d\nu = \frac{1}{2\pi}\int_0^{2\pi}\left(1 + e\cos\nu\right) d\nu = 1$$

Hence $\frac{d\Omega}{dt} = -3\frac{nJ_2 R_e^{\ 2}}{p^2}\cos i$ (6.13)

The process for the two other elements is similar but too lengthy to be reproduced here. The results for all elements are summarised below, where the zero-suffixed values are the initial values:

$$\overline{a} = a_0$$

$$\overline{e} = e_0$$

$$i = i_0$$

$$\overline{\omega} = \omega_0 + \frac{3}{2}\frac{\overline{n} J_2 R_e^{\ 2}}{p^2}\left(2 - \frac{5}{2}\sin^2 i\right) t \qquad (6.14)$$

$$\overline{\Omega} = \Omega_0 - \frac{3}{2}\frac{\overline{n} J_2 R_e^{\ 2}}{p^2}\left(\cos i\right) t$$

$$\overline{M} = M_0 + \overline{n}\, t$$

$$\overline{n} = n_0\left[1 + \frac{3}{2}\frac{J_2 R_e^{\ 2}}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\left(1 - e^2\right)^{1/2}\right]$$

The expression for Ω describes the effect of the precession of the satellite orbit, caused by the slight torque on the satellite by the equatorial bulge of the earth. The minus sign in the expression for Ω indicates that for direct orbits (inclination less than 90 degrees), the nodes move westward, (*regression of the line of nodes*) while for a retrograde orbit (inclination more than 90 degrees), the nodes move eastward. Regression of the line of nodes may be used to achieve a sun-synchronous orbit. This requirement can be satisfied for a retrograde orbit, if $-\frac{3}{2}\frac{\bar{n}J_2R^2}{p^2}(\cos i)T = 2\pi$, where T is the number of seconds in a tropical year.

The expression for ω is equal to zero if i = 63° 26' or i = 116° 34'. This is important for satellite communication systems, which employ highly elliptical orbits, because at those inclinations, the apogee will remain at the same latitude.

6.10 Lunar-Solar Perturbations

The effective attraction of sun or moon on a unit mass near the earth is equal to the attraction by sun or moon at the mass minus the attraction at the earth centre (see Figure 6-1):

$$\mathbf{F} = \mu_p\left(\frac{\mathbf{r}_{sp}}{r_{sp}^{\ 3}} - \frac{\mathbf{r}_p}{r_p^{\ 3}}\right), \tag{6.15}$$

where r_p represents the distance to either sun or moon. The perturbation can also be described in terms of a disturbing potential:

$$R = -\mu_p\left(\frac{1}{r_{sp}} - \frac{r\cos\phi}{r_P^2}\right) \tag{6.16}$$

Since r << r_p for near-earth satellites, expression (15) can be approximated as follows:

$$R = -\mu_p\left[\frac{1}{\sqrt{r_p^{\ 2} - 2rr_p\cos\phi + r^2}} - \frac{r\cos\phi}{r_p^{\ 2}}\right] = -\frac{\mu}{r_p}\left[\frac{1}{\sqrt{1-2\left(\frac{r}{r_p}\right)\cos\phi + \left(\frac{r}{r_p}\right)^2}} - \frac{r\cos\phi}{r_p^{\ 2}}\right] \approx \tag{6.17}$$

$$-\frac{\mu}{r_p}\left[1+\left(\frac{r}{r_p}\right)\cos\phi - \frac{1}{2}\left(\frac{r}{r_p}\right)^2 + \frac{3}{2}\left(\frac{r}{r_p}\right)^2\cos^2\phi - \left(\frac{r}{r_p}\right)\cos\phi + ...\right] = -\frac{\mu r^2}{2r_p^{\ 3}}\left(3\cos^2\phi - 1\right)$$

where we used: $(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - .....$

The similarity between (6.17) and expression (6.11) for earth oblateness indicates that the long term effect of this perturbation will again be a precession of the orbit, in this case about the ecliptic pole for solar perturbations, and about an axis normal to the moon's orbit plane for lunar perturbations. Short and long term effects can be calculated using the Variation of Parameters Method. First substitute (6.14) in (6.7):

$$S = \frac{\partial R}{\partial r} = \frac{\mu r}{r_p^{\ 3}}\left(3\cos^2\phi - 1\right)$$

$$T = \frac{1}{r}\frac{\partial R}{\partial u} = -\frac{\mu r}{r_p^{\ 3}}\left(6\cos\phi\frac{\partial\cos\phi}{\partial u}\right), \tag{6.18}$$

$$W = \frac{1}{r\sin u}\frac{\partial R}{\partial i} = -\frac{\mu r}{r_p^{\ 3}}\left(6\cos\phi\frac{\partial\cos\phi}{\partial i}\right)$$

where $\cos\phi = \dfrac{\mathbf{r} \bullet \mathbf{r}_p}{r.r_p} = \mathbf{r}^{(1)} \bullet \mathbf{r}_p^{(1)}$,

$$\mathbf{r}^{(1)} = \begin{pmatrix} \cos\Omega\cos u - \sin\Omega\sin u\cos i, \\ \sin\Omega\cos u + \cos\Omega\sin u\cos i, \\ \sin i\sin u \end{pmatrix}, \text{and}$$

$\mathbf{r}_{ss}^{(1)} = (\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta)$, where α, δ are the sun's or moon's right ascension and declination respectively.

6.11 Atmospheric Drag

Drag, by definition, will be opposite to the velocity of the satellite relative to the atmosphere. The perturbing acceleration **F** is given by:

$$\mathbf{F} = \frac{1}{2m} C_D A_e \rho v^2, \tag{6.19}$$

where

C_D = aerodynamic drag coefficient
A_e = average cross-sectional area of the satellite,
ρ = air density
m = satellite mass
v = satellite velocity relative to the rotating atmosphere.

The aerodynamic drag coefficient C_D is approximately 1 when the mean free path of the atmospheric molecules is small compared to the satellite size, although the exact value depends upon the satellite shape, the nature of its surfaces, and its attitude. C_D takes a value between 2 and 3 - dependent on the shape of the satellite - when the mean free path is large compared with the dimensions of the satellite. Exact values are best determined by actual flight test, but a value of about 2.2 will give a good, slightly conservative, result.

The complicating factor in the calculation of drag, is the variable nature of the atmospheric density. For heights between 0 and 100km, the U.S. Standard Atmosphere of 1962 can be used. A good approximation is the following simple exponential law:

$$\rho = \rho_o \exp[-h/H], \tag{6.20}$$

where

h = altitude above sea level in km
ρ_o = sea level density at 288.15 K = 1.225 kg/m^3
H = "scale height" = 6.966 km

However, for greater heights, the atmospheric density exhibits variations with respect to altitude and latitude. There are large day-to-night variations, a 27-day cycle due to ultraviolet radiation, and an 11 year cycle due to the solar flux. The table below gives average values for the atmospheric density for three representative values of solar activity (based on the Jacchia 1964 model).

Table 4 Atmospheric Density (kg/m^3) as a function of Altitude

Altitude (km)	Quiet Sun	Average Sun	Active Sun
150	7.4E-10	7.4E-10	7.4E-10
200	1.7E-10	2.4E-10	3.0E-10
250	5.5E-11	1.0E-10	1.3E-10
300	1.7E-11	3.8E-11	5.5E-11
350	5.9E-12	1.7E-11	2.6E-11
400	2.3E-12	7.4E-12	1.3E-11
450	7.4E-13	3.3E-12	6.9E-12
500	3.0E-13	1.7E-12	3.8E-12
550	1.2E-13	8.0E-13	2.1E-12
600	5.7E-14	4.1E-13	1.2E-12
650	2.5E-14	2.3E-13	7.4E-13
700	1.2E-14	1.3E-13	4.8E-13

Figures 6-4 and 6-5 illustrate the influence of atmospheric drag on lifetime of a satellite in low orbit. In both examples a satellite of 450 kg and A_e of 25 m^2 is in a 400 km circular orbit. The first case relates to average solar conditions; the second case is for an active sun (These figures show the results of simulations, using the MatLab Simulink software package).

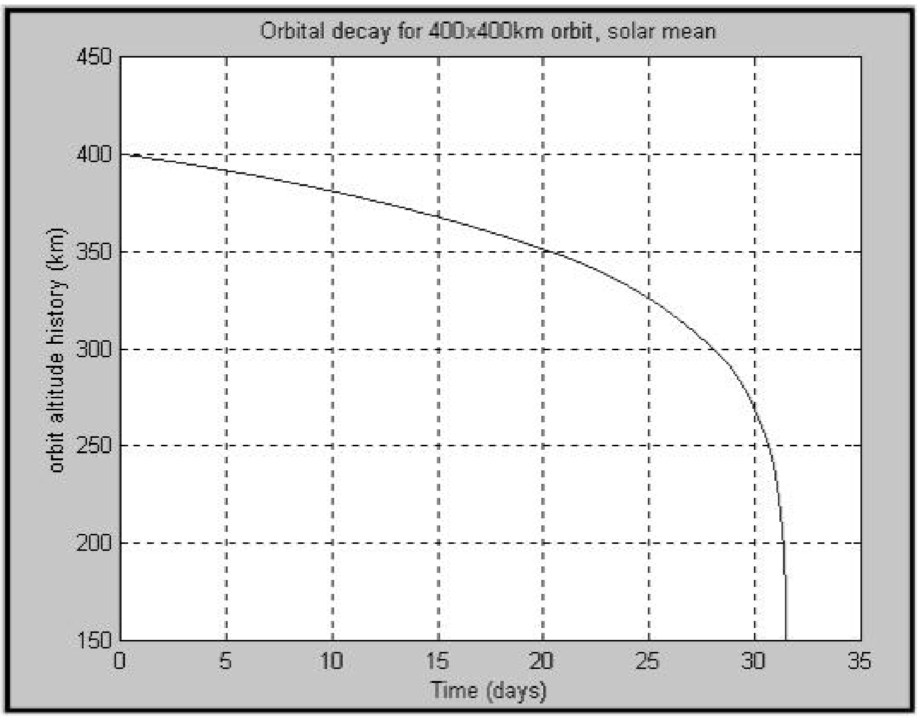


Figure 6-4 Orbit Decay for a 450 kg Satellite; Mean Sun

Figure 6-5 Orbit Decay for a 450 kg Satellite; Active Sun

Before the acceleration **F** in expresion 6.19 can be used in the Variation of Parameters Method, it is necessary to resolve **F** into two orthogonal components **S** and **T**:

$$\mathbf{S} = -\mathbf{F}\sin\beta = -\mathbf{F}\frac{e\sin\nu}{\sqrt{1+2e\cos\nu+e^2}}$$
$$\mathbf{T} = -\mathbf{F}\cos\beta = -\mathbf{F}\frac{1+e\cos\nu}{\sqrt{1+2e\cos\nu+e^2}}, \qquad (6.21)$$

where β is the flight path angle, i.e. the complement of the angle between velocity **v** and **r**.

6.12 Radiation Pressure

Radiation from the sun exerts a force on the satellite, which can not be neglected, especially in the case of three-axis stabilised satellites with large solar arrays. The normal and tangential forces on each surface area element dA of the sunlit part of the satellite are:

$$dF_N = \frac{I}{c}dA(1+\sigma)\cos^2\phi$$
$$dF_T = \frac{I}{c}dA(1-\sigma)\sin\phi\cos\phi, \qquad (6.22)$$

where

$I = 1.36 \times 10^3$ joule/m^2/sec is the solar radiation constant at the mean sun-earth distance,
c is the speed of light, σ is the reflectivity of the surface, and ϕ is the angle between the normal to the surface element and the direction towards the sun. Since satellites have complex geometries, with some portions reflecting diffusely, and other portions specularly, and with changing aspect angles towards the sun, the total resulting force **F** on the satellite will be a

complex function of satellite shape, location and attitude. However, assuming that the surface has a high reflectivity, the resulting acceleration will act along the sun-satellite direction, and can be estimated by the following expression:

$$\mathbf{F} = -2\frac{I}{c}\frac{A_e}{m}\mathbf{r}_{ss}^{(\mathbf{1})}, \tag{6.23}$$

where A_e is the projection of the total sunlit area on a plane perpendicular to the satellite -sun direction, and $\mathbf{r}_{ss}^{(1)}$ is a vector of unit length in the direction satellite-sun. The Variation of Parameters Method requires components **S**, **T**, and **W**. The values of these can be found as follows, see Figure 6-4

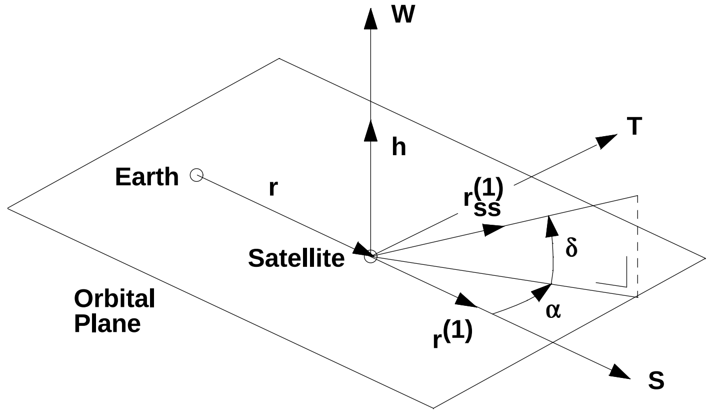


Figure 6-4

$$\begin{aligned} W &= \left(\mathbf{r}_{ss}^{(1)} \bullet \mathbf{h}^{(1)}\right)F \\ S &= \left(\mathbf{r}_{ss}^{(1)} \bullet \mathbf{r}^{(1)}\right)F \quad , \\ T &= \sqrt{F^2 - W^2 - S^2} \end{aligned} \tag{6.24}$$

where $\mathbf{h} = \sqrt{\mu a(1-e^2)}(\sin i \sin\Omega, -\sin i \cos\Omega, \cos i)$ is the angular momentum vector, perpendicular to the orbital plane, and

$\mathbf{r}_{ss}^{(1)} = (\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta)$, and

where α, δ are the sun's right ascension and declination respectively.

6.13 Use of the Variation of Parameters Method to calculate Satellite Position and Velocity

The few analytical expressions that result from the theoretical considerations are insufficient to allow accurate orbit position station keeping. The variation of parameters method is therefore used in practice in conjunction with numerical techniques. The procedure for the calculation of satellite position and velocity in a perturbed orbit, using the expressions for the rate of change of the orbital elements, is summarised below. Assume that the values for the orbital elements are given at the start time t = 0:

Step 1: At t = 0 calculate the satellite position and velocity for the given orbital elements.

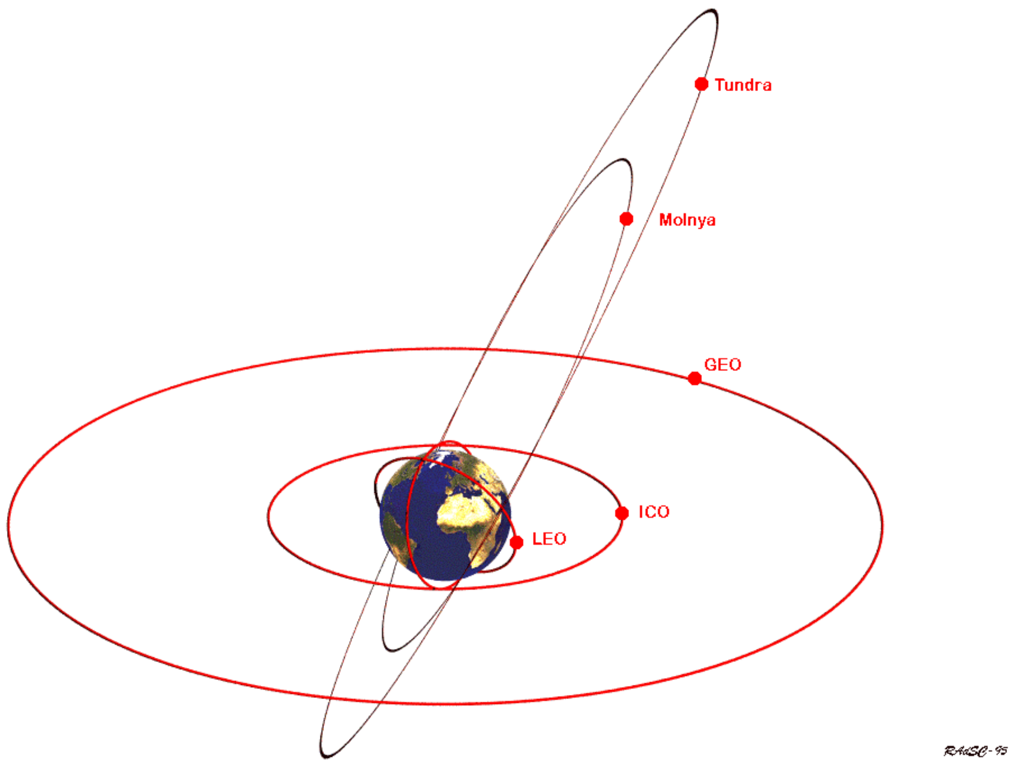
Step 2: At t = 0, using the given position and velocity of the satellite in space, compute the three components S, T, W of the perturbation force.

Step 3: At t = 0, using the given values for the orbital elements, and the radial, transverse, and lateral components of the perturbing accelerations, S, T, W, compute the six rates-of-change of the orbital elements.

Step 4: Numerically integrate the rates-of change over a time interval delta-t (e.g. using a Runge-Kutta method).

Step 5: Determine the values of the orbital elements at the end of the time interval, go back to step 1, and repeat until the end time is reached.

APPENDIX 1 - ORBIT TYPES



1 LOW EARTH ORBIT (LEO)

LEOs are either elliptical or (more usual) circular orbits at a height of less than 2,000 km above the surface of the earth. The orbit *period* at these altitudes varies between ninety minutes and two hours. The radius of the *footprint* of a communications satellite in LEO varies from 3000 to 4000 km. The maximum time during which a satellite in LEO orbit is above the local horizon for an observer on the earth is up to 20 minutes. A global communications system using this type of orbit, requires a large number of satellites, in a number of different, inclined, orbits. When a satellite serving a particular user moves below the local horizon, it needs to be able to hand over the service to a succeeding one in the same or adjacent orbit. Due to the relatively large movement of a satellite in LEO with respect to an observer on the earth, satellite systems using this type of orbit need to be able to cope with large *Doppler shifts*. Satellites in LEO are also affected by *atmospheric drag*, which causes the orbit to gradually deteriorate.

Examples of Big LEO" systems are GlobalstarTM (48+8 satellites in 8 orbital planes at 1400 km), and Iridium (66+6 satellites in 6 orbital planes at 780 km). There are many “Small LEO" systems. One particular example of such a system is PoSat, built by SSTL in1993 and launched into an 822 x 800 km orbit, inclined at 98.6 degrees.

2 MEDIUM EARTH ORBITS (MEO), ALSO CALLED INTERMEDIATE CIRCULAR ORBITS (ICO)

MEOs are circular orbits at an altitude of around 10,000 km. Their orbit *period* measures about 6 hours. The maximum time during which a satellite in LEO orbit is above the local horizon for an observer on the earth is in the order of a few hours. A global communications system using this type of orbit, requires a modest number of satellites in 2 to 3 orbital planes to achieve global coverage. MEO satellites are operated in a similar way to LEO systems. However, compared to a LEO system, hand-over is less frequent, and *propagation delay* and *free space loss* are greater. Examples of Companies employing MEO orbits are ICO (10 +2 satellites in 2 inclined planes at 10390 km), and Odyssey (12 + 3 satellites in 3 inclined planes, at 10355 km).

3 HIGHLY ELLIPTICAL ORBITS (HEO)

HEOs typically have a perigee at about 500 km above the surface of the earth and an *apogee* as high as 50,000 km. The orbits are inclined at 63.4 degrees in order to provide communications services to locations at high northern latitudes. The particular *inclination* value is selected in order to avoid *rotation of the apses*, i.e. the intersection of a line from earth centre to *apogee* and the earth surface will always occur at a latitude of 63.4 degrees North. Orbit *period* varies from eight to 24 hours. Owing to the high *eccentricity* of the orbit, a satellite will spend about two thirds of the orbital *period* near *apogee*, and during that time it appears to be almost stationary for an observer on the earth (this is referred to as *apogee* dwell). After this *period* a switchover needs to occur to another satellite in the same orbit in order to avoid loss of communications. *Free space loss* and *propagation delay* for this type of orbit is comparable to that of geostationary satellites. However, due to the relatively large movement of a satellite in HEO with respect to an observer on the earth, satellite systems using this type of orbit need to be able to cope with large *Doppler shifts*. Examples of HEO systems are:

- the Russian Molniya system, which employs 3 satellites in three 12 hour orbits separated by 120 degrees around the earth, with *apogee* distance at 39,354 km and perigee at 1000 km;
- the Russian Tundra system, which employs 2 satellites in two 24 hour orbits separated by 180 degrees around the earth, with *apogee* distance at 53,622 km and perigee at 17,951 km;
- the proposed Loopus system, which employs 3 satellites in three 8 hour orbits separated by 120 degrees around the earth, with *apogee* distance at 39,117 km and perigee at 1,238 km;
- the European Space Agency's (ESA's) proposed Archimedes system. Archimedes employs a so-called “M-HEO" 8 hour orbit. This produces three *apogee*s spaced at 120 degrees. Each *apogee* corresponds to a service area, which could cover a major population centre, for example the full European continent, the Far East and North America.

4 GEOSYNCHRONOUS ORBIT

A geo-synchronous orbit is any type of orbit, which produces a repeating ground track. This is achieved with an orbit *period*, which is approximately an integer multiple or sub-multiple of a *sidereal day* (NOTE: The word “approximately" is used because there is a need to correct for the *node regression* due to the *precession* of the satellite orbit).

5 GEOSTATIONARY ORBIT (GEO)

A geostationary orbit is a circular prograde orbit in the equatorial plane with an orbital *period* equal to that of the Earth, which is achieved with an orbital radius of 6.6107 (Equatorial) Earth Radii, or an orbital height of 35786 km. A satellite in a geostationary orbit will appear fixed above the surface of the Earth. In practice, the orbit has small non-zero values for *inclination* and *eccentricity*, causing the satellite to trace out a small figure of eight in the sky. The footprint, or service area of a geostationary satellite covers almost 1/3 of the Earth's surface (from about 75 degrees South to about 75 degrees North latitude), so that near-global coverage can be achieved with a minimum of three satellites in orbit. A disadvantage of a geostationary satellite in a voice communication system is the *round -trip delay* of approximately 250 milliseconds.

6 POLAR ORBIT

A polar orbit is inclined at about 90 degrees to the equatorial plane, covering both poles. The orbit is fixed in space, and the Earth rotates underneath. Therefore, a single satellite in a Polar Orbit, provides in principle coverage to the entire globe, although there are long *period*s during which the satellite is out of view of a particular ground station. This may be acceptable for a store-and-forward type of communication system. Accessibility can of course be improved by deploying more than one satellite in different orbital planes.

Most Small-LEO systems employ polar, or near-polar, orbits. A particular example of a system that uses this type of orbit is the COSPAS-SARSAT Maritime Search and Rescue system. This system uses 8 satellites in 8 near polar orbits: Four SARSAT satellites move in 860 km orbits, inclined at 99 degrees, which makes them sun-synchronous. Four COSPAS satellites move in 1000 km orbits, inclined at 82 degrees.

7 SUN-SYNCHRONOUS ORBIT

In a Sun-synchronous or Helio-synchronous orbit, the angle between the orbital plane and Sun remains constant, which results in consistent light conditions of the satellite. This can be achieved by a careful selection of orbital height, *eccentricity* and *inclination* which produces a precession of the orbit (*node rotation*) of approximately one degree eastward each day, equal to the apparent motion of the sun. This condition can only be achieved for a satellite in a *retrograde* orbit. A satellite in sun-synchronous orbit crosses the equator and each latitude at the same time each day. This type of orbit is therefore advantageous for an Earth Observation satellite, as it provides constant lighting conditions.

APPENDIX 2 - BASIC PROPERTIES OF THE ELLIPSE

1 GENERAL

This curve is defined as the locus of points P for which the sum of distances to two fixed points - the focal points - remains constant. In figure 1 we have made the sum of these two distances equal to 2a. The position of the two focal points, F_1 and F_2, is chosen so that both are on the x-axis: F_1 at x = + ae, and F_2 at x = - ae, where e is an as yet arbitrary constant, the **eccentricity**, with a value between 0, and 1. Figure 1 shows the point P at three different positions.

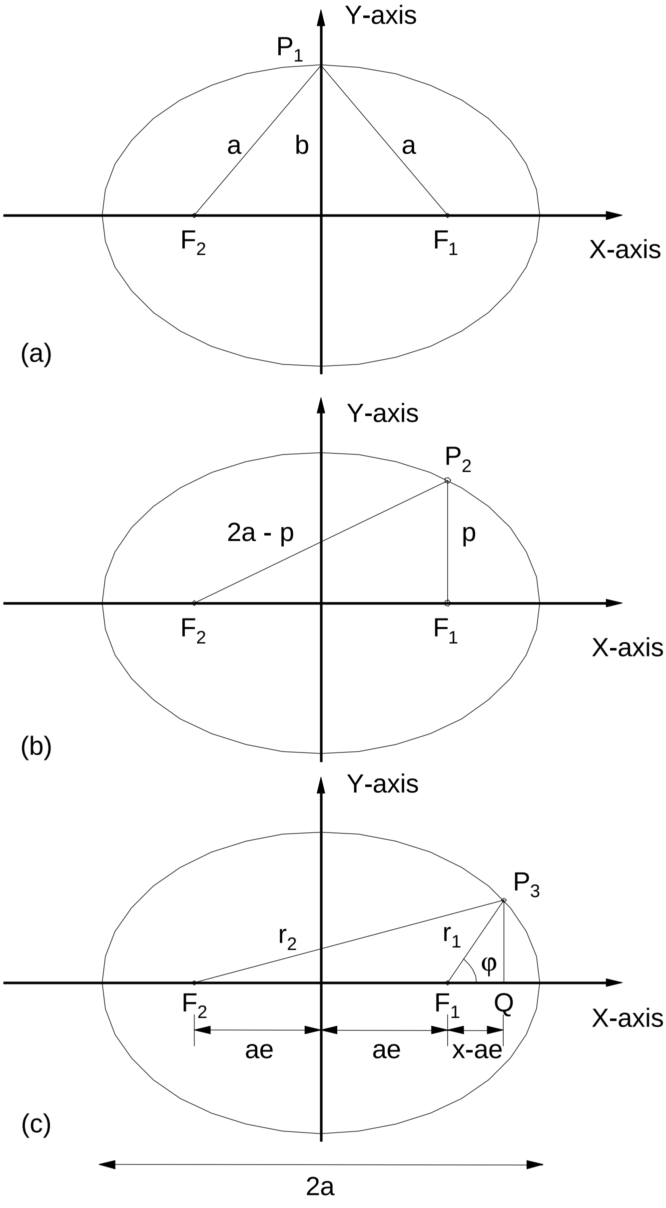


Figure 1-1

With P at P_1 (figure 1-1a), it is clear that the **semi-minor axis** $b = a(1-e^2)^{1/2}$;
With P at P_2 (figure 1-1b), it follows that $p^2 + (2ae)^2=(2a-p)^2$,
or the **semi-latus rectum** $p = a(1-e^2)$;

With P at an arbitrary position P_3 (figure 1-1c), it can be shown that:

$r_1 = a - ex$, and $r_2 = a + ex$.
For proof, apply the cosine rule ($a^2 = b^2 + c^2 - 2bc \cos A$) to triangle $F_2 P_3 F_1$, and use $r_1 + r_2 = 2a$.

2 EQUATION IN CARTESIAN CO-ORDINATES

Apply Pythagoras ' rule to triangle $F_1 P_3 Q$:

$$y^2 + (x + ae)^2 = r_2.^2 = (a + ex)^2,$$

or $$y^2 + x^2(1 - e^2) = a^2(1 - e^2) = b^2,$$

or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is the usual equation in Cartesian co-ordinates.

3 EQUATION IN POLAR CO-ORDINATES

An equation for the ellipse in polar co-ordinates can be derived from (figure 1-1c):

$r_1.\cos\varphi = x - ae$, or $x = r_1.\cos\varphi + ae$
It was shown before that $r_1.= a - ex$,
or $r_1.= a - e(r_1.\cos\varphi + ae)$,
or $r_1.(1 + e.\cos\varphi) = a(1 - e^2) = p$,

It follows that the equation of the ellipse in polar co-ordinates is:

$$r = \frac{p}{1 + e\cos\varphi}$$

4 AREA OF THE ELLIPSE

See figure 2 where a circle of radius a has been circumscribed about the ellipse. P_1 is an arbitrary point on the circle. P_2 is the corresponding point on the ellipse with the same x-co-ordinate. It follows that:

P_1: $$\frac{x^2}{a^2} + \frac{y_{circle}^2}{a^2} = 1$$

P_2: $$\frac{x^2}{a^2}+\frac{y_{ellipse}{}^2}{b^2}=1$$

or $$\frac{y_{ellipse}}{y_{circle}}=\frac{b}{a}$$

It follows that the area of the ellipse is given by:

$$A_{ellipse}=\frac{b}{a}A_{circle}=\frac{b}{a}\pi a^2=\pi ab$$

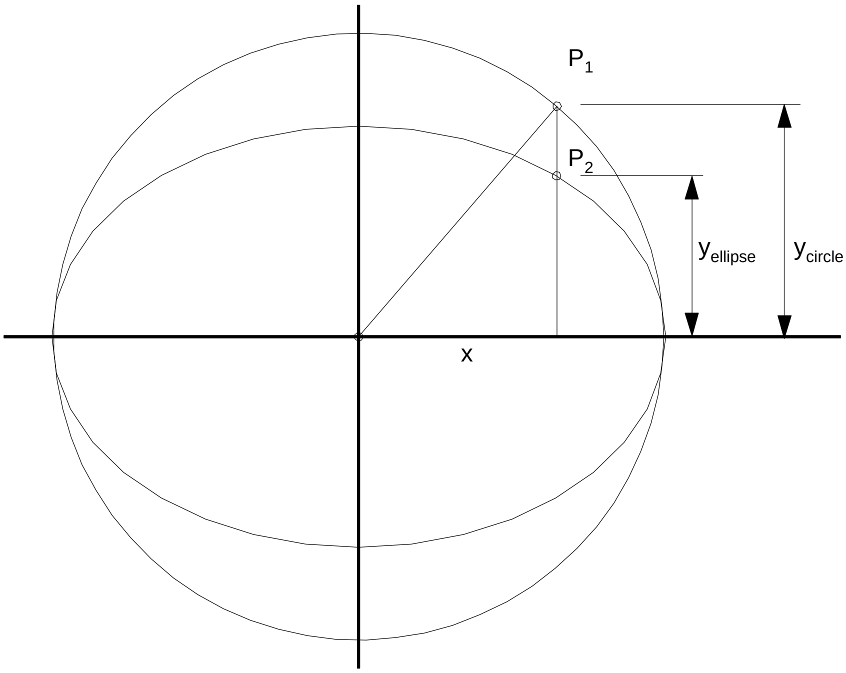


Figure 4-1

APPENDIX 3 - VECTOR ANALYSIS, THE VERY BASICS

1 DEFINITIONS

Scalar: Any quantity that can be specified by a single value, such as mass, speed or length of a vector. The symbol used in most cases will be a single letter in either upper or lower case. An exception is the notation for the magnitude of a vector, see below.

Vector: A quantity having both magnitude and direction, such as force or velocity. The symbol used in most cases will be a single bold-faced letter in either upper or lower case. The magnitude of vector **a** can be shown either as a, or as |**a**|, also called modulus of **a**, or length of **a**.

Unit Vector: A vector having magnitude one, or unit magnitude. Certain letters are more or less reserved to denote unit vectors along the co-ordinate axes in rectangular co-ordinate systems, e.g. **i, j, k** and **p, q, r**. Another notation often used is $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ for the unit vectors along the x, y , and z-axes in a rectangular co-ordinate system.

A symbol, also used for a unit vector having the same direction as the (non-zero) vector **a,** is $\mathbf{a}^{(1)}$. It follows that $\mathbf{a}^{(1)} = \mathbf{a} / |\mathbf{a}|$.

Equal Vectors: Two vectors are said to be equal if their directions are the same, i.e. they are parallel to each other, and their values are equal, regardless of the positions of their origins.

Collinear Vectors: Two vectors are said to be collinear if they are parallel to each other.

Coplanar Vectors: Two vectors are said to be coplanar if they are parallel to the same plane.

2 VECTOR OPERATIONS

2.1 Addition of Vectors

The sum of two vectors **a** and **b** is a vector **c** which can be obtained by constructing a parallelogram OACB, with **a** and **b** as two adjoining sides (Figure 2-1). The sum vector **c** is the diagonal OC.

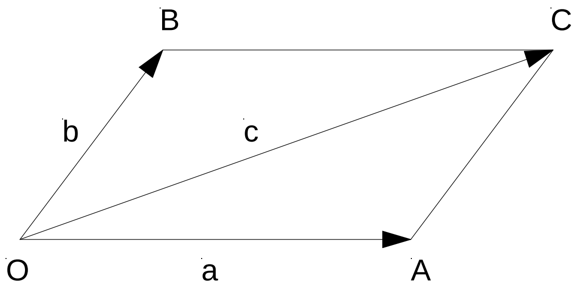


Figure 2-1

It follows that any vector **r** lying in a plane may be resolved into two components, in *any* given directions, which are parallel to the plane. If we choose two rectangular axes parallel to the plane, and **i** and **j** are unit vectors along those axes, **r** can be expressed as **r** = x**i** + y**j** (see Figure 2-2).

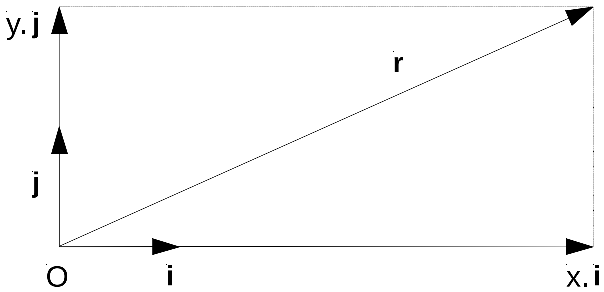


Figure 2-2

Similarly in three dimensional space, a vector **r** can be described as **r** = x**i** + y**j** +z**k**, where **i**, **j**, and **k** are unit vectors along three orthogonal axes.

It follows from this definition that both the **commutative law** and the **associative law** apply to the addition of vectors:

a + **b** = **b** + **a**, (commutative law)
a + (**b** + **c**) = (**a** + **b**) + **c** = **a** + **b** + **c**, (associative law).

The vector **a** + **a** is naturally called 2**a**, and is a vector with the same direction as **a** but of twice its length. Similarly **a** + **a** + **a** + **a** + … = m**a.**

Inspection of figure 2-3 reveals that m(**a** + **b**) = m**a** + m**b**, (distributive law)

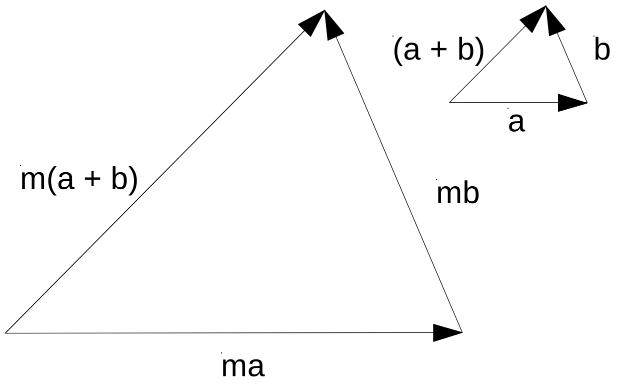


Figure 2-3

2.2 Scalar or Dot Product

The scalar product of **a** and **b**, denoted by **a.b** is a scalar quantity defined by:

a.b = **|a||b|** cos A, where A is the angle between the two vectors **a** and **b**.

It follows that if two vectors are perpendicular, their scalar product is equal to zero since cos A = 0. In particular **i.j =j.k = k.i =** 0.

If two vectors are collinear, **a.b = |a||b|.** In particular **i.i = j.j =k.k** = 1.

From these results we see that if **a** = a_x**i** + a_y**j** + a_z**k**, and **b** = b_x**i** + b_y**j** + b_z**k**, then
a.b = (a_x**i** + a_y**j** + a_z**k**) .(b_x**i** + b_y**j** + b_z**k**) = $a_x b_x$**i.i** + $a_y b_y$**j.j** + $a_z b_z$**k.k** + R,
where R = C_1 **i.j** + C_2 **i.k** + … comprises the remaining terms which are all equal to zero because each term is the internal product of two orthogonal vectors.

Hence **a.b** = $a_x b_x$**i.i** + $a_y b_y$**j.j** + $a_z b_z$**k.k =** $a_x b_x + a_y b_y + a_z b_z$.

It follows that **a.b** = **b.a** (commutative law).

2.3 Vector Product

The vector product of **a** and **b**, denoted by **a*b** is a vector ab sin A $\mathbf{e}_{ab}$, where A is the angle between the directions of the two vectors, and $\mathbf{e}_{ab}$ is a vector of unit length perpendicular to both **a** and **b**, such that **a**, **b** and $\mathbf{e}_{ab}$ form a right handed system, i.e. if we rotate **a** to **b** through the smallest angle possible, $\mathbf{e}_{ab}$ has the same direction as the resulting movement of a corkscrew when rotated through the same angle. This is illustrated in figure 2-4.

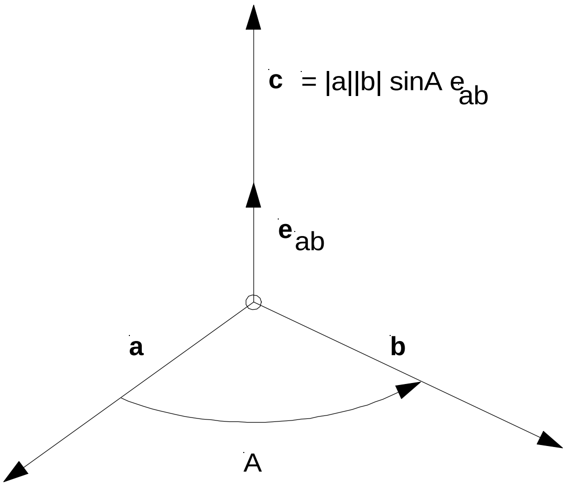


Figure 2-4

Vector cross products also satisfy the distributive and associative laws:

a * (**b** + **c**) = **a** * **b** + **a** * **c**, (distributive law)
m **a***b** = (m**a**)* **b** = **a** * (m**b**), (associative law).

If **a** and **b** are collinear, then **a** * **b** = 0 since sin A = 0. In particular:

a * **a** = 0, and **i** * **i = j** * **j** = **k** * **k** = 0.

It follows from the preceding that:

a * **b** = (a_x**i** + a_y**j** + a_z**k**) * (b_x**i** + b_y**j** + b_z**k**) = ($a_y b_z$ - $a_z b_y$)**i** + ($a_z b_x$ - $a_x b_z$)**j** + ($a_x b_y$ - $a_y b_x$)**k**.

This can also be written as a determinant:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \mathbf{b}_x & \mathbf{b}_y & \mathbf{b}_z \end{vmatrix}$$

2.4 Vector Triple Product

The vector triple product **a * (b * c)** is a vector perpendicular to both **a** and (**b** * **c**). Using the expression for **a * b** given above, it can be shown that **a *** (**b * c**) = (**a.c**)**b** - (**a.b**)**c**.

2.5 Differentiation of vectors

If the co-ordinates a_x, a_y, and a_z of vector **a** are functions of a variable t, then
d**a**/dt = da_x/dt **i** + da_y/dt **j** + da_z/dt **k**. It follows that d**a**/dt is also a vector. Similarly:

d(**a.b**)/dt = d($a_x b_x$ + $a_y b_y$ + $a_z b_z$)/dt = a_x db_x/dt + a_y db_y/dt + a_z db_z/dt + b_x da_x/dt + b_y da_y/dt + b_z da_z/dt = (**a.**d**b**/dt) + (**b.**d**a**/dt), and without proof:

d(**a** * **b**)/dt = (**a *** d**b**/dt) + (d**a**/dt * **b**).

2.6 Velocity

Let a point describe a curve in space. At time t the position of the point is P (figure 1-5).

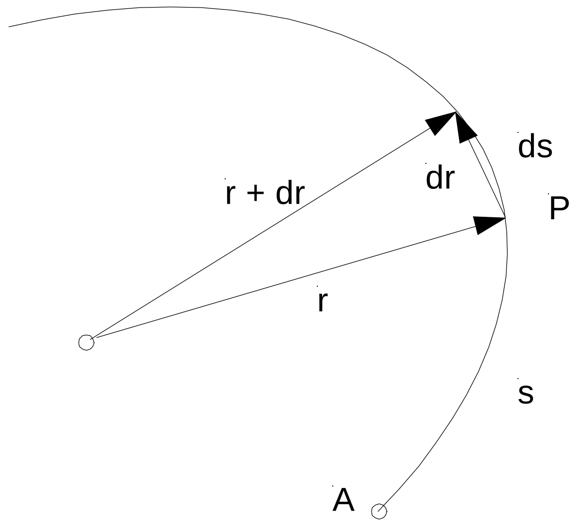


Figure 2-5

Let s be the arc length, measured from some fixed point A on the curve to the point P. The position of P can also be described as the end point of a variable vector **r.** In a time interval dt, the point P has moved a distance ds along the curve, and the vector **r** has become **r** + d**r**. We now define the velocity as the rate of change of the vector **r**. Hence:

v = d**r**/dt = d**r**/ds . ds/dt

It is clear that |d**r**| = ds. Hence **t** = dr/ds is a unit vector, tangent to the curve at point P. The quantity
v = ds/dt is the rate of change of arc length, or the linear velocity.
It follows that **v** = v **t**.

2.7 The Vector Operator ∇

The Vector Operator ∇ (Nabla) is defined as:

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

If V(x,y,z) is a scalar function, then V(x,y,z)=C represents a surface in three-dimensional space, and
$\nabla V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}$, is a vector, the direction of which is along the normal to the surface, and magnitude of which is the gradient of V along the normal. The "rounded" d indicates that while differentiation takes place with respect to one of the variables, the other variables may be treated as constants.

If V is a function of the radius vector only, then it can easily be shown that:

$$\nabla V(r) = \frac{dV(r)}{dr}\mathbf{e}_r$$

APPENDIX 4 - SOME PROPERTIES OF THE ELLIPSOID

1 INTRODUCTION

For accurate calculations of azimuth and elevation to a satellite (e.g. for orbit determination purposes), the actual , non-spherical, shape of the earth needs to be taken into account. The simplest mathematical reference surface is a rotational ellipsoid. This means that the intersection of the equatorial plane and the reference surface is a circle. The intersection of any meridian and the ellipsoid surface is an ellipse, with a half major axis (a) which is often referred to as “equatorial radius”, and a half minor axis (b) which is referred to as “polar radius”. The ratio f = (a - b) / a is called the “flattening” of the earth. The flattening f is related to the eccentricity e by the following expression: $e^2 = f\left(2 - f\right)$.

Several reference ellipsoids are in use in different countries, although the World Geodetic System 1984 (WGS-84) is now a well established universal reference. For WGS-84, a=6,378,137.0 m, and f=1/298.257223563.

The co-ordinates of an earth station antenna can be described in terms of longitude and geodetic latitude on the reference ellipsoid, and height H above the ellipsoid. Figure 1-1 shows a highly exaggerated representation of the shape of the earth. In this figure, P is the earth station location; the line PS is the local vertical in P. The angle φ between PS and the equatorial plane is known as the “geodetic latitude”. Note that the local vertical in P does not go through the centre of the earth. We define the “transverse plane” as the plane which contains the line PS, and which is perpendicular to the meridian plane. It will be proven later that the intersection of the transverse plane and the ellipsoid is an ellipse. The tangent to this curve in P corresponds to the east-west direction in P.

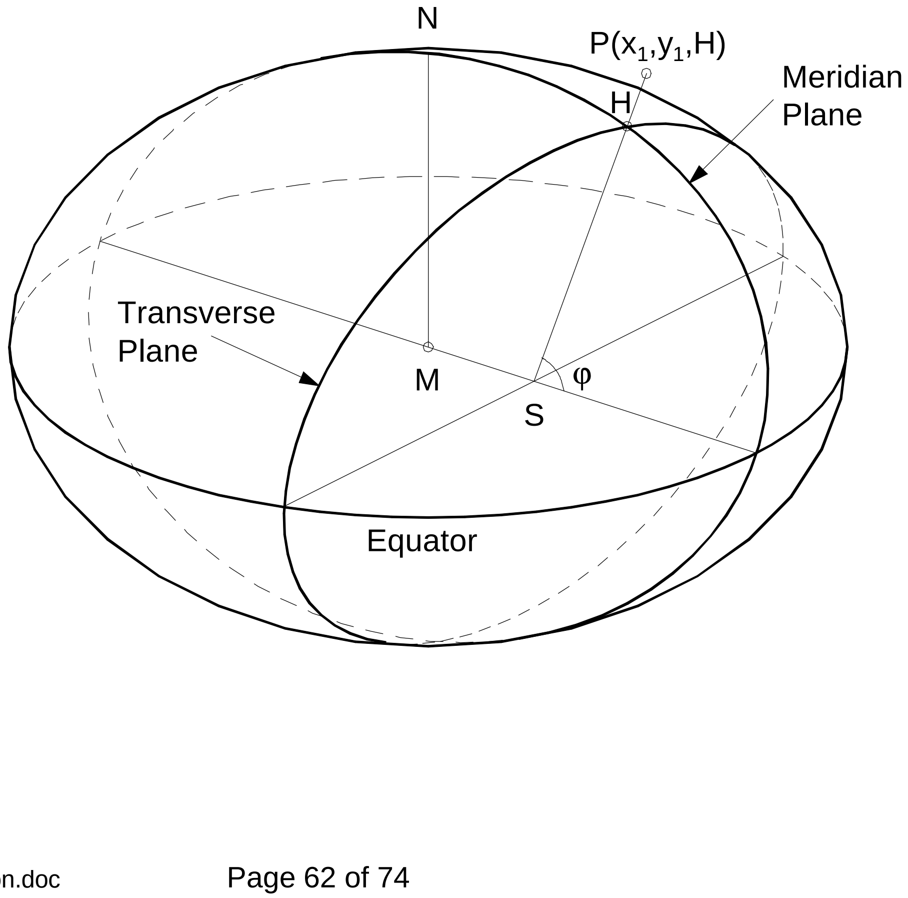


Figure 1-1

2 SOME BASIC CONCEPTS INVOLVING THE GEODETIC LATITUDE

2.1 Derivatives

Figure 2-1 shows part of the meridian plane through P. The x-axis is the intersection of the meridian and the equatorial plane. The y-axis corresponds with the earth axis of rotation. The tangent in P makes an angle θ with the x-axis which can be found by differentiating the expression for the ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \tan\theta=\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y} \tag{1}$$

Since PS is perpendicular to the tangent at P, It follows that $\tan\varphi=\frac{a^2}{b^2}\frac{y}{x}$ (2)

The second derivative is found by differentiating expression (1):

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\frac{y-x\frac{dy}{dx}}{y^2}=-\frac{b^2}{a^2}\frac{y+x\frac{b^2}{a^2}\frac{x}{y}}{y^2}=-\frac{b^4}{a^2}\frac{\frac{y^2}{b^2}+\frac{x^2}{a^2}}{y^3}=-\frac{b^4}{a^2y^3} \tag{3}$$

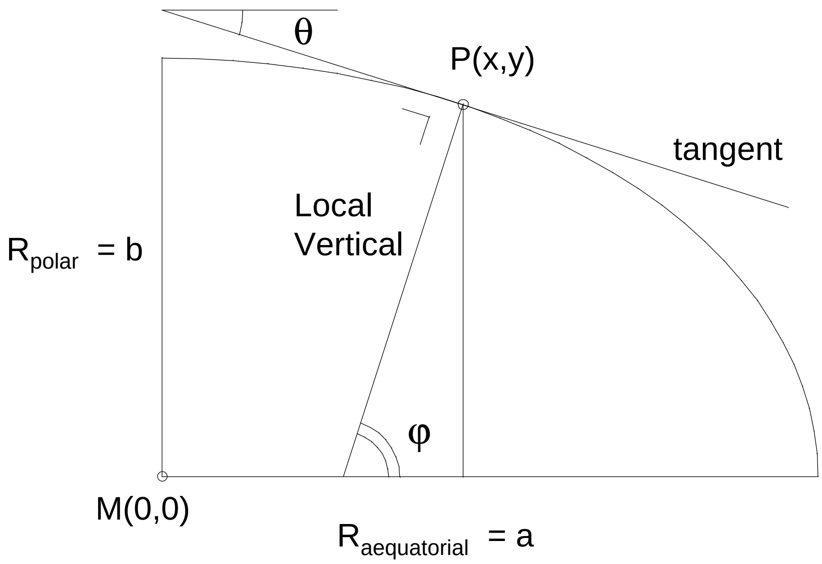


Figure 2-1

2.2 Expressions for x, y and the Derivatives of y = f(x) as Functions of Geodetic Latitude φ

It is often desirable to use the geodetic latitude φ as independent variable in various expressions rather than x and y. It is therefore useful to develop expressions for x and y as functions of φ**.** Start with the equation for the ellipse in x and y, and use expression (2) for φ:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{a^2}\left(1+\frac{a^2}{b^2}\frac{y^2}{x^2}\right)=1 \Rightarrow \frac{x^2}{a^2}\left(1+\frac{b^2}{a^2}\tan^2\varphi\right)=1, \text{ or}$$

$$x = \frac{a}{\left(1 + \frac{b^2}{a^2}\tan^2\varphi\right)^{1/2}} = \frac{a\cos\varphi}{\left(1 - e^2\sin^2\varphi\right)^{1/2}} \tag{4a}$$

Similarly:

$$y = \frac{b}{\left(1 + \frac{a^2}{b^2}\tan^{-2}\varphi\right)^{1/2}} = \frac{b^2}{a}\frac{\sin\varphi}{\left(1 - e^2\sin^2\varphi\right)^{1/2}} = \frac{a\left(1 - e^2\right)\sin\varphi}{\left(1 - e^2\sin^2\varphi\right)^{1/2}} \tag{4b}$$

Inspection of figure 2-1 shows that:

$$\frac{dy}{dx} = -\frac{1}{\tan\varphi} \tag{5a}$$

Substituting (4b) in (3) produces the following expression for d^2y/dx^2 as a function of φ:

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3} = -\frac{b^4}{a^2}\frac{\left(1 - e^2\sin^2\varphi\right)^{3/2}}{a^3\left(1 - e^2\right)^3\sin^3\varphi} = \frac{\left(1 - e^2\sin^2\varphi\right)^{3/2}}{a\left(1 - e^2\right)\sin^3\varphi} \tag{5b}$$

2.3 Equation of the Ellipse. Change of Independent Variable

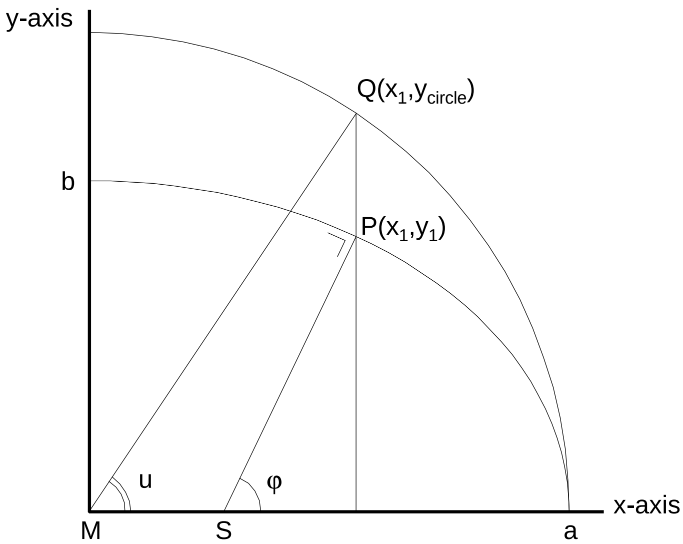
In a later section, an equation for the ellipse is needed with u = a -x as independent variable instead of x. Start with the equation for the ellipse in x and y, and substitute u for x:

$$\frac{(a-u)^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow$$
$$1 - \frac{2au}{a^2} + \frac{u^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y^2 + \frac{u^2}{\left(a^2/b^2\right)} = 2\frac{u}{\left(a/b^2\right)} \tag{6}$$

2.4 Relation between Latitude and "Reduced Latitude"

In some calculations it is useful to introduce an auxiliary variable, called the "reduced latitude", and use this instead of the geodetic latitude. The reduced latitude is defined as the angle u measured from the centre of the ellipse to a point Q on the circumscribed circle which has the same abscissa x1 as the point P on the ellipse (see figure 2-2).

Figure 2-2



Start with expression (2) for $\tan\varphi$, and use $y_1 / y_c = b/a$ to obtain a relation between u and φ:

$$\tan u = \frac{y_{circle}}{x_1} = \frac{a}{b}\frac{y_1}{x_1} = \frac{a}{b}\frac{b^2}{a^2}\tan\varphi = \frac{b}{a}\tan\varphi \tag{7}$$

3 PROOF THAT THE INTERSECTION OF THE TRANSVERSE PLANE AND THE ELLIPSOID IS AN ELLIPSE

Figure 3-1 shows part of the curve which forms the intersection of the transverse plane and the ellipsoid. In the equation for this curve, we use variables u and v measured along two orthogonal axes. The variable u has its origin in P, and is measured along local vertical PS. The other variable (v) is measured along an axis in the transverse plane, perpendicular to PS and parallel to the equatorial plane.

Since point A lies on an ellipse with main axes a and b, we can write:

$$\frac{OA^2}{a^2} + \frac{OM^2}{b^2} = 1 \Rightarrow OA^2 = a^2\left(1 - \frac{OM^2}{b^2}\right)$$

Because of the rotational symmetry of the ellipsoid, OB = OA. Therefore:

$$v^2 = OB^2 - OQ^2 = OA^2 - OQ^2 = a^2\left(1 - \frac{OM^2}{b^2}\right) - OQ^2 \tag{8}$$

Inspection of figure 3-1 shows that:

$$OM = y_1 - u\sin\varphi \tag{9a}$$

$$OQ = x_1 - u\cos\varphi \tag{9b}$$

Substituting (9a) and (9b) into (8), and using expression (4) for x1, produces the following equation in u and v for the unknown curve:

$$v^2 = a^2\left(1 - \frac{(y_1 - u\sin\varphi)^2}{b^2}\right) - (x_1 - u\cos\varphi)^2 =$$

$$-u^2\left(\frac{a^2}{b^2}\sin^2\varphi + \cos^2\varphi\right) + 2u\left(\frac{a^2}{b^2}y_1\sin\varphi + x_1\cos\varphi\right) =$$

$$-u^2\left(\frac{a^2}{b^2}\sin^2\varphi + 1 - \sin^2\varphi\right) + 2u\frac{x_1}{\cos\varphi} =$$

$$-u^2\left(1 + \frac{e^2}{1-e^2}\sin^2\varphi\right) + 2u\frac{a}{(1 - e^2\sin^2\varphi)^{1/2}} \Rightarrow$$

$$v^2 + Au^2 = 2Bu \tag{10}$$

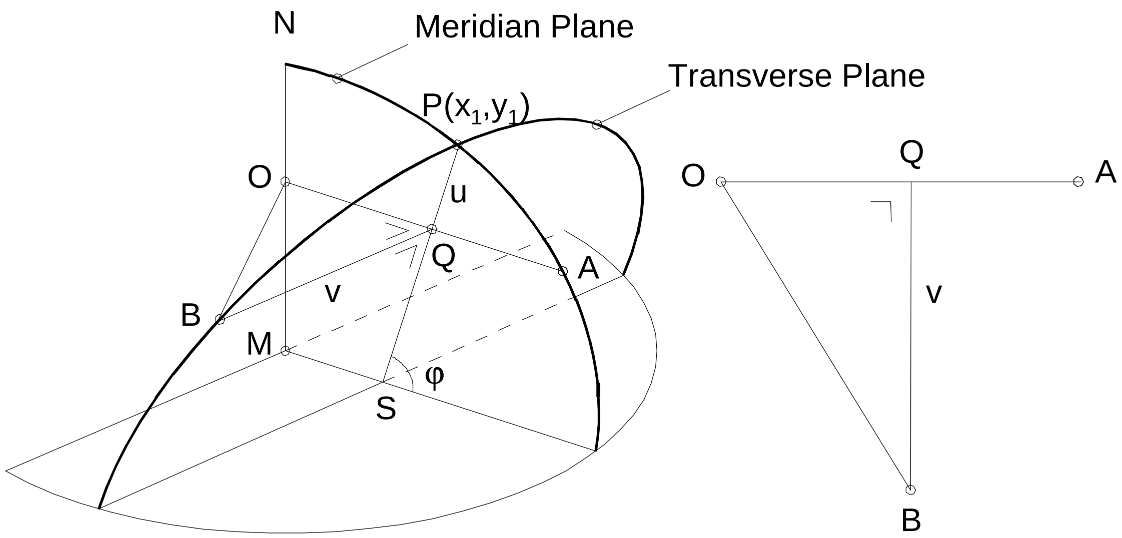


Figure 3-1

Comparing (6) and (10), proves that the unknown curve is an ellipse with main axes a_T and b_T following from:

$$A = \frac{b_T^2}{a_T^2} = \left(1 + \frac{e^2}{1-e^2}\sin^2\varphi\right), \tag{11a}$$

$$B = \frac{b_T^2}{a_T} = \frac{a}{\left(1-e^2\sin^2\varphi\right)^{1/2}} \tag{11b}$$

Therefore:

$$a_T = \frac{B}{A} = \frac{a}{\left(1-e^2\sin^2\varphi\right)^{1/2}\left(1+\frac{e^2}{1-e^2}\sin^2\varphi\right)} \tag{12a}$$

and

$$b_T = \frac{B}{A^{1/2}} = \frac{a}{\left(1+\frac{1}{4}\frac{e^4}{1-e^2}\sin^2 2\varphi\right)^{1/2}} \tag{12b}$$

4 MERIDIAN AND TRANSVERSE RADIUS OF CURVATURE

It should be noted that the equatorial radius and the polar radius are not radii of curvature: Since the earth is flatter at the poles than at the equator, the polar radius of *curvature* must be greater than the equatorial radius of *curvature*. It is obvious that the radius of curvature of the ellipsoid varies with latitude. It is less well known that in any point P on the earth surface, the radius of curvature measured in the meridian plane, i.e. along a north-south direction, is different from that measured in any other direction. In practice we are interested in the radii of curvature in two perpendicular planes, i.e. in the meridian plane, and in the “transverse” plane.

We can now calculate both the meridian, and the transverse radius of curvature, using the expression for the radius of curvature for a plane curve, derived in Annex 1:

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \tag{13}$$

4.1 Meridian Radius of Curvature

Substitute expressions (5a) and (5b) in (13):

$$\rho = \frac{\left(1+\frac{1}{tg^2\varphi}\right)^{3/2} a\left(1-e^2\right)\sin^3\varphi}{\left(1-e^2\sin^2\varphi\right)^{3/2}} = \frac{a\left(1-e^2\right)\left(\frac{\sin^2\varphi+\cos^2\varphi}{\sin^2\varphi}\right)^{3/2}\left(\sin^2\varphi\right)^{3/2}}{\left(1-e^2\sin^2\varphi\right)^{3/2}} = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\varphi\right)^{3/2}}$$

or:

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\varphi\right)^{3/2}} \tag{14}$$

It follows that the meridian radius of curvature at the equator becomes:

$$\rho(\varphi = 0) = a\left(1-e^2\right) = \frac{a^2\left(1-e^2\right)}{a} = \frac{b^2}{a} \tag{15}$$

and the meridian radius of curvature at the poles becomes:

$$\rho(\varphi = \pm\pi) = \frac{a\left(1-e^2\right)}{\left(1-e^2\right)^{3/2}} = \frac{a}{\left(1-e^2\right)^{1/2}} = \frac{a^2}{a\left(1-e^2\right)^{1/2}} = \frac{a^2}{b} \tag{16}$$

4.2 Transverse Radius of Curvature

In order to distinguish between the two radii of curvature, the transverse radius of curvature is usually referred to as ν. Note that point P is at the intersection of the (transverse) ellipse and its major axis. We can therefore use expression (15) for the meridian curvature, and expression (11b) for B to obtain an expression for ν:

$$\nu = \frac{b_T^2}{a_T} = B = a\left(1-e^2\sin^2\varphi\right)^{-1/2} \tag{17}$$

5 SOME FURTHER PROPERTIES OF THE ELLIPSOID

5.1 Centre of Spherical Curvature

Inspection of expressions (4a) for x_1, and (17) for ν shows that the centre of transverse spherical curvature M_t lies on the semi-minor axis (i.e. on the line through the poles), in the opposite hemisphere as point P (see also figure 5-1). Since the meridian radius of curvature is smaller than the transverse radius, it follows that the centre of curvature of the meridian spherical curvature lies somewhere on the line PM_T.

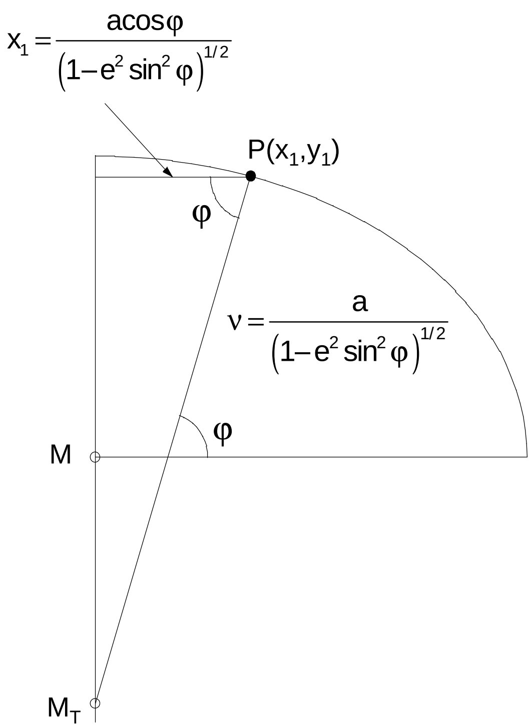


Figure 5-1

5.2 Meridian Arc Distance

The length of the arc measured along the meridian between two latitudes φ_1 and φ_2 is given by the expression:

$$s = \int_{\varphi_1}^{\varphi_2} \rho \, d\varphi = \int_{\varphi_1}^{\varphi_2} \rho \, d\varphi = a(1-e^2)\int_{\varphi_1}^{\varphi_2} \frac{1}{\left(1-e^2\sin^2\varphi\right)^{3/2}} d\varphi \qquad (18)$$

Use the following formulae to evaluate expression (18):

$$\int \sin^2\varphi \, d\varphi = \frac{\varphi}{2} - \frac{\sin\varphi\cos\varphi}{2} \qquad (19a)$$

$$\int \sin^4\varphi \, d\varphi = \frac{3\varphi}{8} - \frac{3\sin\varphi\cos\varphi}{8} - \frac{\sin^3\varphi\cos\varphi}{4} = \frac{3\varphi}{8} - \frac{\sin\varphi\cos\varphi}{2} + \frac{\sin 4\varphi}{32} \qquad (19b)$$

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots\ldots \qquad (19c)$$

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \ldots\ldots \qquad (19d)$$

$$(1+x)^{-3/2} = 1 - \frac{3x}{2} + \frac{15x^2}{8} - \frac{35x^3}{16} + \ldots\ldots 2 \qquad (19e)$$

Therefore:

$$s = \int_{\varphi_1}^{\varphi_2} \rho \, d\varphi = a(1-e^2)\int_{\varphi_1}^{\varphi_2} \frac{1}{\left(1-e^2\sin^2\varphi\right)^{3/2}} d\varphi = a(1-e^2)\int_{\varphi_1}^{\varphi_2}\left(1+\frac{3e^2}{2}\sin^2\varphi + \frac{15e^4}{8}\sin^4\varphi + ...\right)d\varphi =$$

$$a(1-e^2)\left[\varphi + \frac{3e^2}{2}\left(\frac{\varphi}{2} - \frac{\sin\varphi\cos\varphi}{2}\right) + \frac{15e^4}{8}\left(\frac{3\varphi}{8} - \frac{\sin\varphi\cos\varphi}{2} + \frac{\sin 4\varphi}{32}\right) + ...\right]_{\varphi_1}^{\varphi_2} =$$

$$a(1-e^2)\left[A\varphi - B\frac{\sin 2\varphi}{2} + C\frac{\sin 4\varphi}{4} + ...\right]_{\varphi_1}^{\varphi_2}$$

(20a)

where

$$A = 1 + \frac{3e^2}{4} + \frac{45e^4}{64}$$

$$B = \frac{3e^2}{4} + \frac{15e^4}{16} \qquad (20b)$$

$$C = \frac{15e^4}{64}$$

If we write $\varphi_2 = \varphi_1 + \Delta\varphi$, and $\varphi_1 = \varphi$, expression (20a) becomes:

$$s = a\left(1-e^2\right)\left[A\Delta\varphi - B\cos\left(2\varphi + \Delta\varphi\right) + \frac{C}{2}\cos 2\left(2\varphi + \Delta\varphi\right)\sin 4\Delta\varphi + ...\right] \qquad (21)$$

5.3 Transverse Arc Distance

With reference to figure 5-2, using expression (17) for the transverse radius of curvature ν, and series expansion (19d), the transverse arc distance is given by:

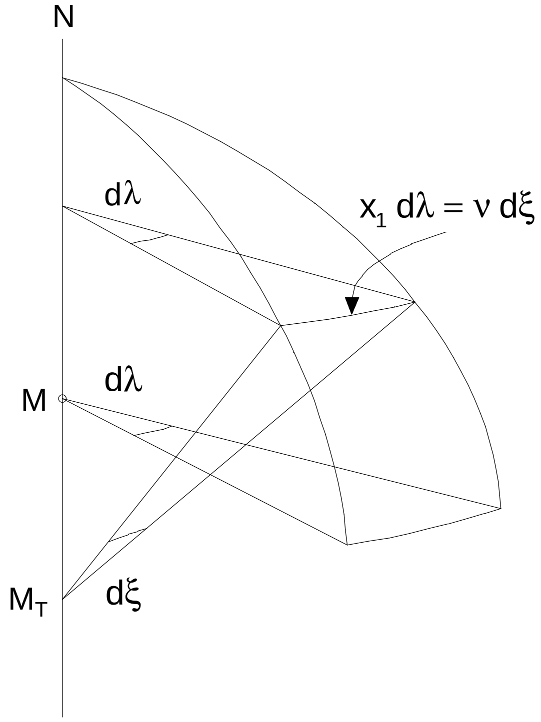


Figure 5-2

$$s = \int_0^{\Delta\xi} \nu d\xi = \int_0^{\Delta\xi} \frac{a}{\left(1-e^2\sin^2\varphi\right)^{1/2}} d\xi = \left(1 - e^2\sin^2\varphi + \frac{3}{8}e^4\sin^4\varphi + ...\right)\Delta\xi \qquad (22)$$

Figure 5-2 shows that $x_1 d\lambda = \nu d\xi$. Therefore, using expression (4a) for x1, and expression (17) for the transverse radius of curvature ν:

$$\Delta\xi = \frac{x_1}{\nu}\Delta\lambda = \frac{a\cos\varphi}{\left(1-e^2\sin^2\varphi\right)^{1/2}}\frac{\left(1-e^2\sin^2\varphi\right)^{1/2}}{a}\Delta\lambda = \cos\varphi\,\Delta\lambda$$

Substitution of this expression for $\Delta\xi$ into (22), produces the following result:

$$s = \left(1-e^2\sin^2\varphi + \frac{3}{8}e^4\sin^4\varphi + ...\right)\cos\varphi\,\Delta\lambda \qquad (23)$$

ANNEX 1 - THE RADIUS OF CURVATURE OF A PLANE CURVE

1 DEFINITIONS OF TANGENT VECTOR (T), NORMAL VECTOR (N), AND RADIUS OF CURVATURE (ρ)

The equation of a plane curve may be written as:

$$x = x(\lambda), \quad y = y(\lambda),$$

where λ is a variable parameter, and x, y are rectangular Cartesian coordinates. It is convenient to choose the arc s measured from some point on the curve as variable parameter, so:

$$\mathbf{r} = \mathbf{r}(s) = x(s)\,\mathbf{i}_x + y(s)\,\mathbf{i}_y$$

Differentiate **r** with respect to s to obtain the following result (see also figure 1):

$$\frac{d\mathbf{r}}{ds} = \frac{d(x\,\mathbf{i}_x + y\,\mathbf{i}_y)}{ds} = \frac{dx}{ds}\mathbf{i}_x + \frac{dy}{ds}\mathbf{i}_y = \cos\alpha\,\mathbf{i}_x + \sin\alpha\,\mathbf{i}_y = \mathbf{t} \qquad (24)$$

The vector **t** is a unit vector, since $\cos^2\alpha + \sin^2\alpha = 1$. Inspection of both expression (24) and figure 1-1 shows that **t** is tangent to the curve.

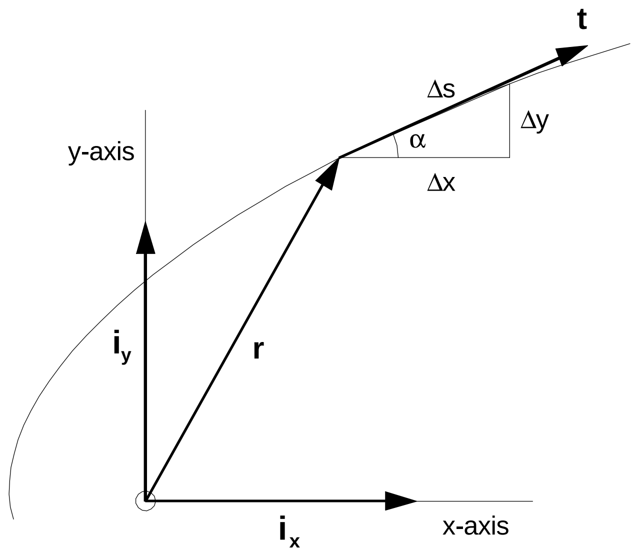


Figure 1-1

The second derivative of **r** with respect to s is a vector, normal to the curve, since:

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{t}}{ds}, \text{ and}$$

$$\mathbf{t}\bullet\mathbf{t} = 1 \quad \Rightarrow \quad \frac{d}{ds}(\mathbf{t}\bullet\mathbf{t}) = 0 \quad \Rightarrow \quad \mathbf{t}\bullet\frac{d\mathbf{t}}{ds} = 0$$

This becomes also clear by inspecting figure 1-2. This figure also shows that the magnitude of the vector $\frac{d\mathbf{t}}{ds} = \frac{1}{\rho}$, where ρ is the radius of curvature of the curve at P. Therefore:

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{1}{\rho}\mathbf{n}, \tag{25}$$

where **n** is the unit vector, normal to the curve.

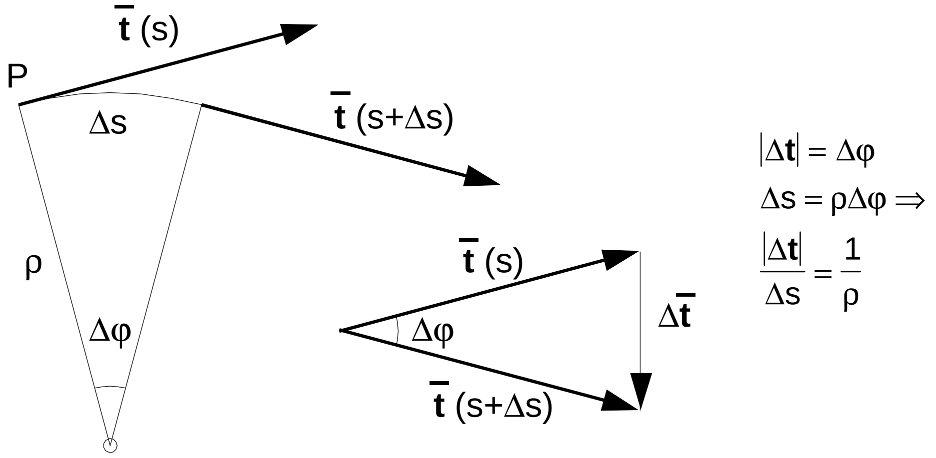


Figure 1-2

2 COORDINATES OF THE NORMAL VECTOR (N)

Figure 2-1 shows the tangent vector **t** and the normal vector **n**. Both vectors are of unit length, and **n** is perpendicular to **t**.

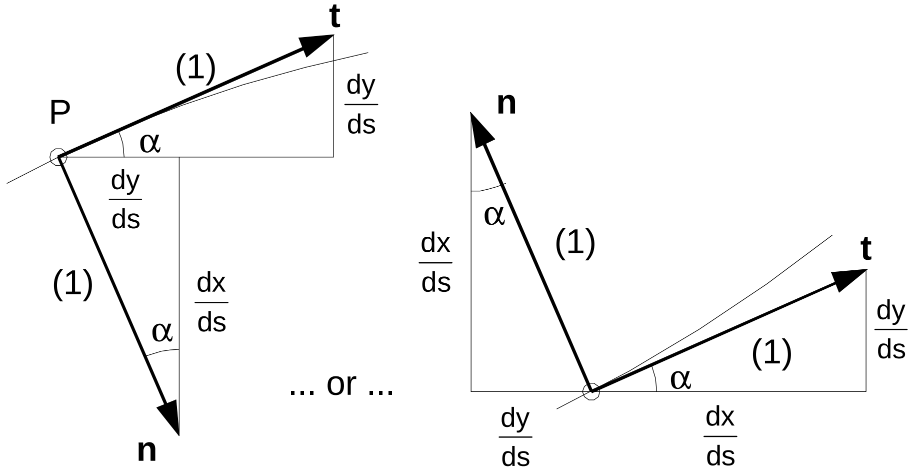


Figure 2-1 (a) (b)

According to expression (24):

$$\mathbf{t} = \frac{dx}{ds}\mathbf{i}_x + \frac{dy}{ds}\mathbf{i}_y$$

Figure 2-1 therefore demonstrates that:

$$\mathbf{n} = \pm\left(\frac{dy}{ds}\mathbf{i}_x - \frac{dx}{ds}\mathbf{i}_y\right) = \pm\left(\frac{dy}{dx}\frac{dx}{ds}\mathbf{i}_x - \frac{dx}{ds}\mathbf{i}_y\right) = \pm\left(\frac{\frac{dy}{dx}\mathbf{i}_x - \mathbf{i}_y}{\frac{ds}{dx}}\right) \tag{26}$$

The sign of **n** in expression (26) depends on the shape of the curve in point P. Therefore, if $\frac{d^2y}{dx^2} < 0$ as in figure 3a, the plus sign applies. Otherwise, the minus sign applies.

3 EXPRESSION FOR THE RADIUS OF CURVATURE (ρ)

Often a curve is defined in terms of x and y where y = f(x). We are therefore interested in finding an expression for ρ with x as independent variable rather than s. In order to achieve this, we first need expressions for $\frac{ds}{dx}$ and $\frac{d^2s}{dx^2}$. Inspection of figure 1-1 shows that:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 \quad \Rightarrow \quad \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2 \quad \Rightarrow \quad \frac{ds}{dx} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} \tag{27}$$

$$\frac{d^2s}{dx^2} = \frac{d}{dx}\left(\frac{ds}{dx}\right) = \frac{1}{2}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{-1/2} \times 2\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{\frac{ds}{dx}} \tag{28}$$

Expressions (26), (27) and (28) enable us to rewrite (25) as:

$$\frac{1}{\rho}\mathbf{n} = \frac{d^2\mathbf{r}}{dr^2} = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dx}\frac{dx}{ds} = \frac{d}{dx}\left(\frac{d\mathbf{r}}{ds}\right)\frac{1}{\left(\frac{ds}{dx}\right)} = \frac{d}{dx}\left(\frac{\left(\frac{d\mathbf{r}}{dx}\right)}{\left(\frac{ds}{dx}\right)}\right)\frac{1}{\left(\frac{ds}{dx}\right)} = \frac{\frac{ds}{dx}\frac{d^2\mathbf{r}}{dx^2} - \frac{d\mathbf{r}}{dx}\frac{d^2s}{dx^2}}{\left(\frac{ds}{dx}\right)^2}\frac{1}{\left(\frac{ds}{dx}\right)} =$$

$$\frac{\frac{ds}{dx}\frac{d^2\mathbf{r}}{dx^2} - \frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{\frac{ds}{dx}}\frac{d\mathbf{r}}{dx}}{\left(\frac{ds}{dx}\right)^3} = \frac{\left(\frac{ds}{dx}\right)^2\frac{d^2\mathbf{r}}{dx^2} - \frac{dy}{dx}\frac{d^2y}{dx^2}\frac{d\mathbf{r}}{dx}}{\left(\frac{ds}{dx}\right)^4} = \frac{\left(\frac{ds}{dx}\right)^2\frac{d^2}{dx^2}\left(x\mathbf{i}_x + y\mathbf{i}_y\right) - \frac{dy}{dx}\frac{d^2y}{dx^2}\frac{d}{dx}\left(x\mathbf{i}_x + y\mathbf{i}_y\right)}{\left(\frac{ds}{dx}\right)^4} =$$

$$\frac{\left(\frac{ds}{dx}\right)^2\frac{d^2y}{dx^2}\mathbf{i}_y - \frac{dy}{dx}\frac{d^2y}{dx^2}\left(\mathbf{i}_x + \frac{dy}{dx}\mathbf{i}_y\right)}{\left(\frac{ds}{dx}\right)^4} = \frac{\frac{d^2y}{dx^2}\left[-\frac{dy}{dx}\mathbf{i}_x + \left(\left(\frac{ds}{dx}\right)^2 - \left(\frac{dy}{dx}\right)^2\right)\mathbf{i}_y\right]}{\left(\frac{ds}{dx}\right)^4} =$$

$$= \frac{\frac{d^2y}{dx^2}\left[-\frac{dy}{dx}\mathbf{i}_x + \left(1+\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right)^2\right)\mathbf{i}_y\right]}{\left(\frac{ds}{dx}\right)^4} = \frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^3}\frac{\left(-\frac{dy}{dx}\mathbf{i}_x + \mathbf{i}_y\right)}{\left(\frac{ds}{dx}\right)} = \frac{\pm\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}\mathbf{n}$$

Therefore:

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} \qquad (29)$$